Downloaded from rsta.royalsocietypublishing.org

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Radiative Gravitational Fields in General Relativity I. General Structure of the Field outside the Source

L. Blanchet and T. Damour

Phil. Trans. R. Soc. Lond. A 1986 **320**, 379-430
doi: 10.1098/rsta.1986.0125

Email alerting service Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click **here**

To subscribe to *Phil. Trans. R. Soc. Lond. A* go to: **http://rsta.royalsocietypublishing.org/subscriptions**

This journal is © 1986 The Royal Society

Downloaded from rsta.royalsocietypublishing.org

Phil. Trans. R. Soc. Lond. A **320**, 379–430 (1986) [379]
Printed in Great Britain

RADIATIVE GRAVITATIONAL FIELDS IN GENERAL RELATIVITY
I. GENERAL STRUCTURE OF THE FIELD OUTSIDE THE SOURCE

By L. BLANCHET and T. DAMOUR

Groupe d'Astrophysique Relativiste, C.N.R.S. – Observatoire de Paris-Meudon,
92195 *Meudon Principal Cedex, France*
and
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125, *U.S.A.*

(*Communicated by B. Carter, F.R.S. – Received* 8 *July* 1985)

CONTENTS

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to
Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences.
www.jstor.org

Downloaded from rsta.royalsocietypublishing.org

We present a well-defined formal framework, together with appropriate mathematical tools, which allow us to implement in a constructive way, and to investigate in full mathematical details, the Bonnor–Thorne approach to gravitational radiation theory. We show how to construct, within this framework, the general radiative (formal) solution of the Einstein vacuum equations, in harmonic coordinates, which is both past-stationary and past-asymptotically Minkowskian. We investigate the structure of the latter general radiative metric (including all tails and nonlinear effects) both in the near zone and in the far zone. As a side result it is proven that post-Newtonian expansions must be done by using the gauge functions $(\lg c)^p/c^n$ (p, n = positive integers).

1. Introduction

1.1 *Motivations*

Gravitational radiation theory, in the context of general relativity, has been the subject of extensive research, especially during the last twenty-five years. However, it must be admitted that some of the main problems of gravitational radiation theory have not yet received satisfactory answers. In this article and its sequels, we shall consider three of these problems, that we summarize in the following three questions.

Problem 1 ('asymptotic problem'): what is the asymptotic behaviour, appropriate to isolated systems and consistent with Einstein's field equations, of radiative gravitational fields far away from their sources?

Problem 2 ('generation problem'): what is the link between the preceding asymptotic behaviour and the structure and motion of the sources that generate the gravitational radiation?

Problem 3 ('radiation reaction problem'): what is the back-reaction of the emission of gravitational radiation on the source?

These problems were first tackled by Einstein (1916, 1918) by means of a linearized post-Minkowskian approach. Many authors subsequently questioned the validity of the conclusions of these and related pioneering works on the grounds that the nonlinear structure of the field equations might qualitatively change the results of the linearized approach. On the basis of the work of Einstein *et al.* (1938) several authors even doubted the existence of gravitational radiation (see, for example, Infeld & Plebanski 1960). However, the discovery

Downloaded from rsta.royalsocietypublishing.org

of a number of exact wave-like solutions of the vacuum field equations, together with the study of gravitational wave fronts and a thorough investigation of the algebraical and differential properties of the Riemann tensor gave some plausibility to the existence of gravitational radiation (for lucid surveys of these approaches to gravitational radiation and references to the numerous original works see Pirani 1962*a*, *b*). Further important steps were contained in the work of Fock (1959) who emphasized that problem 1 above had to be split into two sub-problems concerning, on the one hand the asymptotic behaviour of the field at very large distances from the source and at very large times in the past, where one should impose a 'no incoming radiation' condition; and on the other hand, the asymptotic behaviour at very large distances and at very large times in the future where nonlinear effects introduce (in harmonic coordinates) divergent logarithmic deviations from the expected linearized behaviour of the 'outgoing radiation'. Fock also pointed out that a possible method for trying to answer problem 2 above involved the use of some kind of matching between a gravitational field computed in a region exterior to the source and another gravitational field determined in a region including the source. Another important step was taken by Bonnor (1959) who introduced a new method of approximation based on the simultaneous use of an expansion in powers of the mass (m) and radius (a) of the source, and who proved that at the nonlinear approximation of order m^2a^4 there was a secular decrease of a certain coefficient of the metric, which reduced to the Schwarzschild mass in the stationary case. This decrease was in perfect agreement with the 'quadrupole energy loss' formula of the linearized theory. Shortly afterwards Bondi *et al.* (1962) and Sachs (1962) introduced a new approach to gravitational radiation theory based on the use of a special type of coordinate system that avoids the appearance of logarithms, and on a different approximation procedure. Instead of assuming that the metric admits an asymptotic expansion in the coupling constant G ('nonlinearity expansion') they assumed the existence of an asymptotic expansion in inverse powers of the (luminosity or affine) distance (r). (The first step of this approach had been taken earlier by Trautman 1958.) They proved that this assumption was not inconsistent with Einstein's vacuum field equations, in the sense that they could construct formal series in powers of r^{-1} that were formal solutions of the latter equations. Some of the important results of this work were the proof that a certain coefficient of the metric, which reduces to the Schwarzschild mass in the stationary case, is monotonically decreasing and a new formulation of the asymptotic behaviour of the 'outgoing' radiative gravitational fields ('peeling behaviour') (Sachs 1961; Newman & Penrose 1962). The approach of Bondi & Sachs was clarified by the geometrical 'conformal' reformulation of Penrose (1963, 1965). The latter conformal approach allowed the weakening of the assumptions used by Bondi & Sachs, and led to many further developments (for reviews see, for example, Geroch 1977; Schmidt 1979; Ashtekar 1984).

However, it should be stressed that, despite its elegance, the whole Bondi–Sachs–Penrose approach to asymptotic structure appears at present to be unsatisfactory for several reasons. Indeed, although it provides an elaborate conceptual framework allowing one to prove theorems and to perform calculations, it rests on a set of assumptions that have not been shown to be satisfied by a sufficiently general solution of the inhomogeneous Einstein field equations. In other words, one can say (Schmidt 1979) that the Bondi–Sachs–Penrose approach provides only a *definition* of a class of space-times that one would like to associate to radiative isolated systems (asymptotically simple space-times with sufficiently smooth past and future null infinities and with zero radiation fields at past null infinity), and that neither the global consistency nor the physical appropriateness of this definition have been proven. There are

Downloaded from rsta.royalsocietypublishing.org

some interesting examples of radiative space–times admitting at least a piece of future null infinity (Schmidt 1981; Ashtekar & Dray 1981; Bičak *et al.* 1983), and recent general theorems of Friedrich (1983 *a*, *b*) showing the *local* consistency of the Bondi–Sachs–Penrose definition with Einstein's *vacuum* field equations, but these results fall short of proving the global consistency of the definition with a generic solution of the inhomogeneous field equations. On the contrary, perturbation calculations have given some hints of inconsistency between the Bondi–Sachs–Penrose definition and some approximate solutions of the field equations (Bardeen & Press 1973; Schmidt & Stewart 1979; Walker & Will 1979; Porrill & Stewart 1981; Damour 1985). A second class of reasons that make the Bondi–Sachs–Penrose approach unsatisfactory is that, although it provides at least a tentative answer to problem 1 above, it seems to be ill-suited for giving answers to problems 2 and 3. Indeed it gives information on the gravitational field only in the form of an asymptotic expansion when $r \to \infty$, which seems *a priori* difficult to relate to the source located within $r \leqslant a$.

Furthermore, the present development of gravitational wave detectors, and the observation of astrophysical systems where gravitational radiation reaction effects may have been or seem to be important (like runaway or binary pulsars) make it urgent to find at least approximate (but reliable) answers to problems 2 and 3 above. Several different approaches aimed at answering the latter problems have been proposed: some are analytical, some are half analytical–half numerical (e.g. perturbations around curved backgrounds), and some are numerical. We shall discuss here only the analytical approaches. Among these, there exist two main classes: the post-Newtonian approaches ($1/c$ expansions) and the post-Minkowskian approaches (G expansions). The post-Newtonian approaches are fraught with serious internal consistency problems because they often lead, in higher approximations, to divergent integrals; this is well known for radiation reaction calculations, see, for example, Kerlick (1980), but is also easily seen to be true in the post-Newtonian wave-generation formalism of Epstein & Wagoner (1975) (see, however, the improved post-Newtonian approaches of Persides 1971; Winicour 1983; Futamase & Schutz 1983; Schäfer 1985). The post-Minkowskian approaches have not shown any signs of internal inconsistency but, because of computational difficulties, they have given answers to problems 2 (Kovács & Thorne 1977) and 3 (Damour 1983 *a*, *b*) only in the special case of a source made of widely separated objects (treated as some kind of point masses). For more general sources the straightforward post-Minkowskian method seems rather powerless. Fortunately there exists another approach that can extend the reach of the post-Minkowskian expansion method to more general sources. This approach (Bonnor 1959; Bonnor & Rotenberg 1966; Couch *et al.* 1968; Hunter & Rotenberg 1969; Thorne 1977, 1980, 1983) combines a post-Minkowskian (PM) expansion (nonlinearity expansion, or asymptotic expansion in powers of Gm) with a multipolar (M) expansion (expansion in irreducible representations of the rotation group in Thorne's formalism, or, equivalently, expansion in powers of the source radius a in Bonnor's formalism). We shall below call it a MPM expansion (Multipolar–post-Minkowskian).

The Bonnor–Thorne approach has already been used to investigate several aspects of gravitational radiation theory (Bonnor & Rotenberg 1966; Hunter & Rotenberg 1969; Bonnor 1974; Thorne 1980; Schumaker & Thorne 1983). However, it must be admitted that the whole Bonnor–Thorne approach still lacks a precise technical framework implementing its ideas in a formally clear way and showing how they lead to a well-defined approximation procedure for solving the field equations to all orders of nonlinearity. The first aim of the present work is to put forward such a clear formal framework, and to define, within it, a constructive *algorithm*

Downloaded from rsta.royalsocietypublishing.org

giving the (formal) general radiative solution of the vacuum field equations (taking into account all orders of nonlinearity). (An outline of this framework has been given in Blanchet & Damour 1984*a*.)

From a practical point of view, one of the main advantages of the Bonnor–Thorne approach over that of Bondi–Sachs is that its spatial and temporal domain of *a priori* validity is expected to be larger. Indeed, the Bondi–Sachs approach, being an asymptotic expansion in inverse powers of r for fixed retarded time $u \sim t - r/c$,† is *a priori* expected to yield a 'good' (i.e. uniform) approximation of an actual radiative metric only in the *far wave zone* $r \gg \lambda$ (λ being a typical wavelength). On the other hand the Bonnor–Thorne approach, being mainly a nonlinearity expansion, is *a priori* expected to yield a good approximation everywhere in the *weak-field* zone $r \gg Gm/c^2$ (this assumes that the simultaneous multipolar expansion is in fact convergent, instead of asymptotic, and thereby does not restrict the domain of validity of the method beyond the fact that one must stay outside the source: $r > a \gtrsim Gm/c^2$). Now, for many sources we shall have $\lambda \gtrsim Gm/c^2$, indeed this is true for slow-motion sources where $\lambda \gg a \gtrsim Gm/c^2$, and several numerical calculations have shown that this stays true even for some strongly relativistic sources (gravitational collapse). Therefore the Bonnor–Thorne approach, covering *a priori* a larger domain, both in space and in time, allows the investigation of more aspects of gravitational radiation theory than the Bondi–Sachs approach; for instance, gravitational wave tails (Hunter & Rotenberg 1969; Couch *et al.* 1968; Bonnor 1974) or the link between the far wave zone and the transition zone ($r \sim \lambda$) or even (for slow-motion sources) the near zone ($r \ll \lambda$) (with the possible consequence of devising a wave-generation formalism, see Thorne 1980). In the present article we shall limit ourselves to investigating the structure of a general radiative metric in the far zone and (for slow sources) in the near zone. In subsequent articles we shall use our general formal framework to investigate the connection between the Bonnor–Thorne and the Bondi–Sachs–Penrose approaches, and to study the links between the far-zone field, the transition-zone field, the near-zone field and the source (thereby extending and putting on a firmer formal basis several results of Thorne 1980). This will provide approximate answers to problems 1, 2 and 3 above (for preliminary results, obtained within this framework, concerning radiation reaction effects beyond the quadrupole approximation see Blanchet & Damour 1984*b*).

1.2. *Assumptions*

In this paper we shall consider what we shall call Multipolar–post-Minkowskian expandable metrics (in short MPM metrics), i.e. *formal series* in powers of Newton's constant G‡

$$\mathfrak{g}^{\alpha\beta}(x^\mu) := \sqrt{g}\, g^{\alpha\beta} = f^{\alpha\beta} + Gh_1^{\alpha\beta} + G^2 h_2^{\alpha\beta} + \ldots + G^n\, h_n^{\alpha\beta} + \ldots, \tag{1.1}$$

such that each term of the series $h_n^{\alpha\beta}(x^0, x^1, x^2, x^3)$ admits a *finite* multipolar expansion associated with the O(3) group of rotations of the spatial coordinates (which leave invariant $r := ((x^1)^2 + (x^2)^2 + (x^3)^2)^{\frac{1}{2}}$ and $t := x^0/c$), i.e.

$$h_n^{\alpha\beta}(x^\mu) = \sum_{l=0}^{l_{\max}} h_{nL}^{\alpha\beta}(r, t)\, \hat{n}^L(\Theta, \Phi), \tag{1.2}$$

† In this paper the symbol $\sim$ is used to represent 'of the order of'.

‡ Notations: signature $-+++$; greek indices $= 0, 1, 2, 3$; latin indices $= 1, 2, 3$; $g := -\det(g_{\mu\nu})$; $f_{\alpha\beta} = f^{\alpha\beta} =$ flat metric $= \mathrm{diag}(-1, +1, +1, +1)$; $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ are the usual sets of non-negative integers, integers, real numbers and complex numbers; $C^p(\mathrm{U})$ is the set of p-times continuously differentiable functions in the open set U ($p \leqslant +\infty$). See other notations at the beginning of Appendix A.

Downloaded from rsta.royalsocietypublishing.org

where $l_{\max}$ is some maximum value of l (depending on n), and where L denotes the multi-index $i_1 i_2 \ldots i_l$, $n^L := n^{i_1} n^{i_2} \ldots n^{i_l}$ with $n^i := x^i/r$ (latin indices $= 1, 2, 3$), and where $\hat{n}^L$ denotes the (symmetric)-trace-free part of n^L. The sum appearing in the right-hand side of (1.2) is equivalent to a finite expansion in usual spherical harmonics: $Y_l^m(\Theta, \Phi)$ (for a discussion of this point and of the link between the 'orbital' expansion (1.2) and a fully irreducible tensor spherical harmonics expansion see Thorne (1980) and Appendix A of this article). We restrict our attention to *finite* multipolar expansions because this will allow us to prove rigorously many results concerning $h_n^{\alpha\beta}$ without having to make strong assumptions about the convergence of multipolar expansions. This way of proceeding is essentially equivalent to the 'double (formal) series' approach of Bonnor (1959) ($\mathscr{g} = f + \sum_p \sum_q m^p a^q h_{pq}$), which leads to consider at each step only a finite number of values of p and q i.e. a finite order in G ($m^p = (Gm)^p$) and a finite multipolar expansion ($\sum_q = \sum_l$). It is, however, hoped that at the end of the construction of $\mathscr{g}$ it will be possible to take the limit of an infinite number of multipoles.

We wish to investigate when such MPM metrics satisfy formally (i.e. in the sense of formal series) the Einstein equations, which read outside the source

$$R_{\alpha\beta}(\mathscr{g}^{\mu\nu}(x^\lambda)) = 0. \tag{1.3}$$

In the present paper we shall impose (in the sense of formal series) three more restrictions on $\mathscr{g}(x)$. First, we shall use harmonic coordinates:

$$\partial_\beta \mathscr{g}^{\alpha\beta}(x^\mu) = 0. \tag{1.4}$$

Second, we shall assume that the metric was stationary in the past, i.e. that there exists a time $-T$ such that

$$(t \leqslant -T) \Rightarrow (\partial/\partial t\, \mathscr{g}^{\alpha\beta}(x^i, t) = 0). \tag{1.5}$$

Third, we shall assume that before the time $-T$ the metric was asymptotically Minkowskian in the weak sense that

$$(t \leqslant -T) \Rightarrow (\lim_{r \to \infty} (\mathscr{g}^{\alpha\beta}(x^i, t)) = f^{\alpha\beta}). \tag{1.6}$$

All conditions (1.1)–(1.6) are assumed to hold in some open domain D of $\mathbb{R}^4$ of the type $r > r_0$ with $r_0 \geqslant 0$ (in fact $r_0 \geqslant a =$ source radius). Among the assumptions (1.1)–(1.6) some will be common to our sequel papers ((1.1)–(1.2)) but we leave open the possibility to relax the auxiliary conditions (1.3)–(1.6) by considering in further works non-harmonic coordinates, always radiating sources (taking the limit $T \to +\infty$), etc...

The plan of this paper is as follows; in §2 we discuss the first step of the method: $h_1^{\alpha\beta}$, that is the general solution of the linearized vacuum Einstein equations; in §3 we present some mathematical tools which will be necessary to deal with the nonlinear higher steps $h_n^{\alpha\beta}$ ($n \geqslant 2$); in §4 we show how to construct algorithmically the general $h_n^{\alpha\beta}$; in §5 we investigate the near-zone structure of the precedingly constructed general radiative metric $\mathscr{g}$; in §6 and at the beginning of §7 we present some further mathematical tools which allow us to investigate (in §7) the far-zone structure of a general radiative MPM metric. Some of the technical details are relegated to the Appendixes.

Downloaded from rsta.royalsocietypublishing.org

2. General past-stationary solution of the linearized vacuum equations

The general past-stationary solution of the linearized harmonic Einstein vacuum equations is not only the first step of our approach, but also will constantly be used in the higher steps of our recursive analysis of the general solution of the nonlinear vacuum equations. Therefore, although several authors (notably Sachs & Bergmann 1958; Sachs 1961; Pirani 1964; Thorne 1980) have already dealt with the linearized solution, the precise conditions of validity of their formal treatments are often unclear, so we wish to start afresh and to present a rigorous, self-contained, derivation of this solution within the assumptions (1.1)–(1.6).

Let D be an open domain of $\mathbb{R}^4$ defined as $\{(\boldsymbol{x}, t) \mid r > r_0\}$ for some $r_0 \geqslant 0$. According to the assumptions (1.1)–(1.6) the problem is to find the most general $h_1^{\alpha\beta}(x^i, t)$ satisfying in D

$$h_1^{\alpha\beta}(x^i, t) = \sum_{l=0}^{l_{\max}} \hat{n}^L h_{1L}^{\alpha\beta}(r, t), \tag{2.1}$$

$$\Box h_1^{\alpha\beta}(x^i, t) = 0, \tag{2.2}$$

$$\partial_\beta h_1^{\alpha\beta}(x^i, t) = 0, \tag{2.3}$$

$$t \leqslant -T \Rightarrow \frac{\partial}{\partial t} h_1^{\alpha\beta}(x^i, t) = 0, \tag{2.4}$$

$$t \leqslant -T \Rightarrow \lim_{r\to\infty} h_1^{\alpha\beta}(x^i, t) = 0, \tag{2.5}$$

where $\Box := f^{\alpha\beta}\partial_{\alpha\beta} = \Delta - c^{-2}\,\partial^2/\partial t^2$.

Let us first notice that the restriction in (2.1) to having only a finite multipolar expansion is mainly a matter of convenience when dealing with the linearized approximation. Indeed, if one assumes only that $h_1^{\alpha\beta}(x^\mu)$ is of class C^2 in the open domain D, then $h_1^{\alpha\beta}(x^\mu)$ can be expanded in an absolutely point-wise convergent multipole series (see Appendix B), each term of which, $h_{1L}^{\alpha\beta}(r, t)$, can be computed as the following integral over the unit sphere $n^i n^i = 1$:

$$h_{1L}^{\alpha\beta}(r, t) = \frac{(2l+1)!!}{4\pi l!}\int \mathrm{d}\Omega(n)\, \hat{n}^L h_1^{\alpha\beta}(rn^i, t) \tag{2.6}$$

(where $(2l+1)!! := (2l+1)\,.\,(2l-1)\ldots 3.1$ and $\mathrm{d}\Omega(n) = \sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\Phi$). Then without assuming any further conditions one can apply the projection operator appearing in (2.6) to (2.2)–(2.5), thereby deducing the same equations for $h_{1L}^{\alpha\beta}(r, t)$ as can be obtained by simply replacing the finite sum (2.1) into (2.2)–(2.5).

Thus, let us start by looking for the most general $C^2(D)$ solution of (2.2)–(2.5) (we shall come back to the more restricted assumption (2.1) only after (2.25)). Let us first investigate the consequences of (2.2). One obtains

$$[\partial_r^2 - (1/c^2)\,\partial_t^2 + (2/r)\,\partial_r - l(l+1)/r^2]\, h_{1L}^{\alpha\beta}(r, t) = 0. \tag{2.7}$$

Let us introduce the usual (Minkowskian) retarded and advanced time variables

$$u := t - r/c, \tag{2.8a}$$

$$v := t + r/c, \tag{2.8b}$$

Downloaded from rsta.royalsocietypublishing.org

and define
$$f(u,v) := (v-u)^{-l} \,.\, h^{\alpha\beta}_{1L}(r,t) \tag{2.9}$$

(we suppress the indices on f for the sake of brevity). Equation (2.7) is then equivalent, in the domain D, to
$$[(v-u)\,\partial_{uv} + (l+1)\,\partial_u - (l+1)\,\partial_v] f = 0. \tag{2.10}$$

The latter equation is a particular case of the Euler–Poisson–Darboux equation $(E_{m,n})$
$$E_{m,n}(f) := (v-u)\,\partial_{uv} f + m\,\partial_u f - n\,\partial_v f = 0. \tag{2.11}$$

It is easily seen that (assuming f sufficiently differentiable)
$$\partial_u E_{m,n}(f) = E_{m,n+1}(\partial_u f), \tag{2.12}$$

therefore if f is a solution of $E_{m,n}$, then $\partial_u f$ is a solution of $E_{m,n+1}$. Darboux (1889) has shown that the converse is also true if $n \neq 0$; that is, if g is a solution of $E_{m,n+1}$, with $n \neq 0$, then there exists a solution f of $E_{m,n}$ such that $g = \partial_u f$ (beware of the incomplete treatment of Copson (1975)). Therefore if one knows the general solution f of $E_{m,n}$, with $n \neq 0$, then $\partial_u f$ is the general solution of $E_{m,n+1}$. Exchanging the roles of u and v, and of m and n, leads to the knowledge of the general solution of $E_{m+1,n} : \partial_v f$ knowing the general solution of $E_{m,n}$ (with $m \neq 0$) $: f$. Now
$$E_{1,1}(f) = (v-u)\,\partial_{uv} f + \partial_u f - \partial_v f = \partial_{uv}[(v-u) f], \tag{2.13}$$

therefore the general solution of $E_{1,1}$ is
$$f_{1,1} = \frac{U(u)+V(v)}{v-u}, \tag{2.14}$$

where U and V are arbitrary functions of one real variable which are at least of class C^1.

Hence, by the preceding argument, the general solution of $E_{l+1,\, l+1}$, with $l \in \mathbb{N}$, which is precisely the equation to be solved, (2.10), is
$$f = \frac{2}{l!\,c^{l+1}} \frac{\partial^{2l}}{\partial u^l\,\partial v^l}\left[\frac{U(u)+V(v)}{v-u}\right], \tag{2.15}$$

where U and V are arbitrary (C^{l+1}) functions of one real variable and where the factor $2/(l!\,c^{l+1})$ has been introduced for later convenience. From (2.9) we then get the general solution for $h^{\alpha\beta}_{1L}(r,t)$:
$$h^{\alpha\beta}_{1L} = \frac{2}{l!\,c^{l+1}} (v-u)^l \frac{\partial^{2l}}{\partial u^l\,\partial v^l}\left(\frac{U^{\alpha\beta}_L(u)+V^{\alpha\beta}_L(v)}{v-u}\right), \tag{2.16}$$

(where $U^{\alpha\beta}_L$ and $V^{\alpha\beta}_L$ have to be, in fact, in $C^{l+2}(\mathbb{R})$ for $h^{\alpha\beta}_1$ to be $C^2(D)$). Expanding the right-hand side of (2.16) by means of the Leibniz formula, and going back to the r, t variables, we obtain
$$h^{\alpha\beta}_{1L} = \frac{(-)^l}{2^l} \cdot \sum_{j=0}^{l} \frac{2^j (2l-j)!}{j!(l-j)!} \cdot \frac{{}^{(j)}U^{\alpha\beta}_L(t-r/c) + (-)^j\, {}^{(j)}V^{\alpha\beta}_L(t+r/c)}{c^j\, r^{l-j+1}}, \tag{2.17}$$

where ${}^{(j)}U(x) = \mathrm{d}^j U/\mathrm{d}x^j$. Thanks to formula (A 35*a*) of Appendix A, the result (2.17) can be re-expressed as
$$\hat{n}^L h^{\alpha\beta}_{1L} = \hat{\partial}_L\left[\frac{U^{\alpha\beta}_L(t-r/c) + V^{\alpha\beta}_L(t+r/c)}{r}\right], \tag{2.18}$$

where $\hat{\partial}_L$ denotes the tracefree part of $\partial_{i_1 i_2 \ldots i_l} = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_l}$ (see Appendix A). Up to now we have obtained the general solution of (2.2) alone. At this point let us impose (2.4) (past-stationarity). From (2.18) and from (A 30), which says that $\hat{\partial}_L$ is proportional to

Downloaded from rsta.royalsocietypublishing.org

$(r^{-1}\partial/\partial r)^l = 2^l(\partial/\partial(r^2))^l$, we see immediately that, when $t \leqslant -T$, $\partial_t\, U_L^{\alpha\beta}(t-r/c) + \partial_t\, V_L^{\alpha\beta}(t+r/c)$ must be an odd polynomial in r of maximum degree $2l-1$, with coefficients depending on t. Writing that the latter polynomial must be cancelled by the operator $\partial_r^2 - \partial_t^2$, we find that there exist $2l$ constants C_j $(2 \leqslant j \leqslant 2l+1$; indices suppressed) such that, when $t \leqslant -T$,

$$\partial_t\, U_L^{\alpha\beta}(t-r/c) + \partial_t\, V_L^{\alpha\beta}(t+r/c) = \sum_{j=2}^{2l+1} jC_j\,[(t-r/c)^{j-1} - (t+r/c)^{j-1}]. \tag{2.19}$$

Separating $\partial_t U$ from $\partial_t V$ and integrating, we find that there exist $2l+3$ constants A, B, $C_j (1 \leqslant j \leqslant 2l+1)$ such that, when $t \leqslant -T$,

$$U_L^{\alpha\beta}(t-r/c) = A + \sum_{j=1}^{2l+1} C_j(t-r/c)^j, \tag{2.20}$$

$$V_L^{\alpha\beta}(t+r/c) = B - \sum_{j=1}^{2l+1} C_j(t+r/c)^j. \tag{2.21}$$

From (A 33) and (A 36) we see that, when $t \leqslant -T$,

$$\hat{n}^L\, h_{1L}^{\alpha\beta} = \hat{\partial}_L\,[(A+B)/r - 2(C_{2l+1}/c^{2l+1})\, r^{2l}]. \tag{2.22}$$

Now, if we impose (2.5) (past-asymptotic Minkowski behaviour) we find that C_{2l+1} must be zero. Moreover, as, even when t is restricted to be anterior to $-T$, the advanced time $t+r/c$ can take any real value, (2.21) will give the value of $V_L^{\alpha\beta}$ all over the definition domain of $h_1^{\alpha\beta}$. Taking into account $C_{2l+1} = 0$ and the formula (A 36), we will not change the value of $h_{1L}^{\alpha\beta}$ if we replace everywhere the function $(t, r) \to V_L^{\alpha\beta}(t+r/c)$ by the function

$$(t, r) \to V_L^{\alpha\beta}(t-r/c) = B - \sum_{j=1}^{2l} C_j(t-r/c)^j \tag{2.23}$$

(note the change from advanced to retarded time). Keeping the notation $U_L^{\alpha\beta}$ for the sum $U_L^{\alpha\beta}(t-r/c) + V_L^{\alpha\beta}(t-r/c)$, we conclude that the general $C^2(D)$ solution of (2.2), (2.4) and (2.5) can be represented in D as an absolutely point-wise convergent multipolar series of the form

$$h_1^{\alpha\beta}(\boldsymbol{x}, t) = \sum_l \hat{\partial}_L \left(\frac{U_L^{\alpha\beta}(t-r/c)}{r}\right), \tag{2.24}$$

where each $U_L^{\alpha\beta}(u)$ (the old U plus V) is a function of class C^{l+2} which becomes a constant $(= A+B$ from (2.20) and (2.23)) when $u \leqslant -T$. Reciprocally, the representation (2.24) will satisfy the requirements (2.2), (2.4) and (2.5) if we choose some Us constant for $u \leqslant -T$ and such that the series converges point-wise to a function of class $C^2(D)$.

To impose the remaining 'harmonicity condition' (2.3), it is convenient to follow Thorne (1980) and to algebraically decompose the objects $U_L^{\alpha\beta}$. From (2.24) it is clear that $U_L^{\alpha\beta}$ can be chosen to be symmetric and trace-free with respect to the indices L: $U_L^{\alpha\beta} = U_{\langle L\rangle}^{\alpha\beta}$ where the brackets $\langle\ \rangle$ denote the 'symmetric–trace-free' part (see Appendix A). Now if we consider the objects U_L^{00}, U_L^{0i}, U_L^{ij} from the point of view of their transformation properties under the O(3) group of spatial rotations (which preserve (2.1)–(2.5)) they bear both 'spin' indices (none, i, ij; spin $s \leqslant 2$) and 'orbital angular momentum' indices $(\langle L\rangle = \langle i_1 i_2 \ldots i_l\rangle$; orbital angular momentum $= l$). Then it is convenient, following Thorne (1980), to perform an 'addition' of 'spin' and 'orbital angular momentum' indices $(\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L})$, i.e. to decompose $U_L^{\alpha\beta}$ into ten

Downloaded from rsta.royalsocietypublishing.org

irreducible algebraic pieces constructed by means of the Kronecker tensor (δ_{ij}), the Levi–Civita (pseudo) tensor (ϵ_{ijk}) and ten symmetric–trace-free (abbreviated as STF) Cartesian tensors $\hat{T}_{(1)J}$, $\hat{T}_{(2)J}$, ..., $\hat{T}_{(10)J}$ ($J = i_1 \ldots i_j$ with $|l-2| \leqslant j \leqslant l+2$) (see Appendix A for further explanations and references). Replacing this decomposition into (2.24) and converting the derivative operators to their usual (reducible) form ∂_L we must do some reshuffling because of the traces of ∂_L (by using $\Delta(r^{-1}F(t-r/c)) = c^{-2}r^{-1}\,\partial_t^2 F(t-r/c)$ in D). Finally we obtain the following representation of $h_1^{\alpha\beta}$ by means of ten STF tensors $A_J(u)$, $B_J(u)$, ..., $J_J(u)$ (skipping the hats on them) which are sufficiently differentiable functions of the retarded time $u = t-r/c$, and which are all constant if $u \leqslant -T$ (this follows from the same property of $U(u)$ and the fact that $A(u)$, ..., $J(u)$ are constructed algebraically from $U(u)$ and $\partial_u^2\, U(u)$):

$$h_1^{00} = \sum_{l\geqslant 0} \partial_L(r^{-1}A_L(u)), \tag{2.25a}$$

$$h_1^{0i} = \sum_{l\geqslant 0} \partial_{iL}(r^{-1}B_L(u)) + \sum_{l\geqslant 1} \{\partial_{L-1}(r^{-1}C_{iL-1}(u)) + \epsilon_{iab}\,\partial_{aL-1}(r^{-1}D_{bL-1}(u))\}, \tag{2.25b}$$

$$\begin{aligned} h_1^{ij} = &\sum_{l\geqslant 0} \{\partial_{ijL}(r^{-1}E_L(u)) + \delta_{ij}\,\partial_L(r^{-1}F_L(u))\} \\ &+ \sum_{l\geqslant 1} \{\partial_{L-1(i}(r^{-1}G_{j)L-1}(u)) + \epsilon_{ab(i}\,\partial_{j)aL-1}(r^{-1}H_{bL-1}(u))\} \\ &+ \sum_{l\geqslant 2} \{\partial_{L-2}(r^{-1}I_{ijL-2}(u)) + \partial_{aL-2}(r^{-1}\epsilon_{ab(i}\,J_{j)bL-2}(u))\}, \end{aligned} \tag{2.25c}$$

where $L-1$ denotes the multi-index $i_1 i_2 \ldots i_{l-1}$, $L-2 := i_1 \ldots i_{l-2}$, and the parentheses denote symmetrization: $T_{(ij)} := (\frac{1}{2})\,(T_{ij} + T_{ji})$. Note that if the multipolar series appearing in (2.24) is truncated there is no problem for reshuffling the terms of (2.25) in an arbitrary manner, but if it is an infinite series the convergence of the algebraically decomposed series (2.25) is (*a priori*) implied by the convergence of (2.24) only if one always keep together the algebraic sub-pieces having (for instance) the same total number of indices on the STF tensors (which is l in (2.25) and which corresponds to what was noted above j in $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$) (see Appendix B). As we are going, in the following, to play separately with these sub-pieces we shall now go back to our initial assumption (1.2) or (2.1) restricting our consideration to truncated multipolar expansions.

Let us now impose to (2.25) the 'harmonicity condition' (2.3). As the decomposition (2.25) is easily checked to be unique (although not orthogonal, which causes the problems of convergence evoked above) it is easy to deduce from the 'harmonicity condition' (2.3) algebraic and differential (because of $\Delta(r^{-1}F(u)) = c^{-2}r^{-1}\,\partial_u^2 F(u)$) constraints among the As, Bs, ... and Js. To express simply these constraints let us define the following new STF tensors:

$$(l \geqslant 0)\ M_L(u) := A_L + 2\,{}^{(1)}B_L + {}^{(2)}E_L + F_L, \tag{2.26a}$$

$$(l \geqslant 1)\ S_L(u) := -D_L - \tfrac{1}{2}\,{}^{(1)}H_L, \tag{2.26b}$$

$$(l \geqslant 0)\ W_L(u) := B_L + \tfrac{1}{2}\,{}^{(1)}E_L, \tag{2.26c}$$

$$(l \geqslant 0)\ X_L(u) := \tfrac{1}{2}E_L, \tag{2.26d}$$

$$(l \geqslant 0)\ Y_L(u) := -{}^{(1)}B_L - {}^{(2)}E_L - F_L, \tag{2.26e}$$

$$(l \geqslant 1)\ Z_L(u) := \tfrac{1}{2}H_L, \tag{2.26f}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Downloaded from rsta.royalsocietypublishing.org

where ${}^{(n)}F(u)$ denotes $\mathrm{d}^nF/\mathrm{d}u^n$. It is easily seen ((2.29) below) that there is a one to one relationship between the set $\{A_L, B_L, D_L, E_L, F_L, H_L\}$ and the new set $\{M_L, S_L, W_L, X_L, Y_L, Z_L\}$. Then we can express the 'harmonicity constraints' in terms of the Ms, Ss, Ws, Xs, Ys and Zs, together with the old Cs, Gs, Is and Js. We find $Y = 0$,

$$ {}^{(1)}M = 0, \tag{2.27a} $$

$$ {}^{(2)}M_i = 0, \tag{2.27b} $$

$$ {}^{(1)}S_i = 0, \tag{2.27c} $$

and

$$ C_L = -{}^{(1)}M_L - {}^{(1)}Y_L, \tag{2.28a} $$

$$ G_L = 2Y_L, \tag{2.28b} $$

$$ I_L = {}^{(2)}M_L, \tag{2.28c} $$

$$ J_L = 2\,{}^{(1)}S_L. \tag{2.28d} $$

Equations (2.28), together with the 'inverse' of (2.26), i.e.

$$ A_L = M_L - {}^{(1)}W_L + {}^{(2)}X_L + Y_L, \tag{2.29a} $$

$$ B_L = W_L - {}^{(1)}X_L, \tag{2.29b} $$

$$ D_L = -S_L - {}^{(1)}Z_L, \tag{2.29c} $$

$$ E_L = 2X_L, \tag{2.29d} $$

$$ F_L = -{}^{(1)}W_L - {}^{(2)}X_L - Y_L, \tag{2.29e} $$

$$ H_L = 2Z_L, \tag{2.29f} $$

show that the general harmonic $h_1^{\alpha\beta}$ can be expressed uniquely in terms of the Ms, Ss, Ws, Xs, Ys and Zs and that these variables must satisfy (2.27) (and $Y = 0$). Apart from the latter equations, the only other constraints that the Ms, ..., Zs have to satisfy are that all of them must become constant when $u \leqslant -T$. This implies from (2.27) that not only $M(u)$ and $S_i(u)$ but also $M_i(u)$ have to be always ($\forall u$) constant. Replacing (2.28) and (2.29) into (2.25) leads to an explicit representation of $h_1^{\alpha\beta}$ in terms of unconstrained quantities. For later convenience, let us replace our 'old' M_L and S_L by some 'new' quantities:

$$ M_L^{\text{new}} = -\tfrac{1}{4}c^2 l!(-)^l\, M_L^{\text{old}}, \tag{2.30a} $$

$$ S_L^{\text{new}} = -\tfrac{1}{4}c^3[(l+1)!/l]\,(-)^l\, S_L^{\text{old}}, \tag{2.30b} $$

and let us use the letter M to symbolize the set of STF tensors $\{M_L^{\text{new}}(u), S_L^{\text{new}}(u)\}$, the letter W to symbolize the set $\{W_L(u), X_L(u), Y_L(u), Z_L(u)\}$ and the square brackets [ ] to denote a functional dependence. We then get:

$$ h_1^{\alpha\beta}[M, W] = h_{\text{can}\,1}^{\alpha\beta}[M] + \partial^\alpha w^\beta[W] + \partial^\beta w^\alpha[W] - f^{\alpha\beta}\partial_\mu w^\mu[W] \tag{2.31} $$

with (dropping the superscript 'new' on M_L and S_L)

$$ h_{\text{can}\,1}^{00}[M] = -\frac{4}{c^2}\sum_{l\geqslant 0}\frac{(-)^l}{l!}\,\partial_L(r^{-1}M_L(t-r/c)), \tag{2.32a} $$

Downloaded from rsta.royalsocietypublishing.org

$$h^{0i}_{\mathrm{can}\,1}[M] = +\frac{4}{c^3}\sum_{l\geqslant 1}\frac{(-)^l}{l!}\,\partial_{L-1}(r^{-1}\,{}^{(1)}M_{iL-1}(t-r/c))$$
$$+\frac{4}{c^3}\sum_{l\geqslant 1}\frac{(-)^l\,l}{(l+1)!}\,\epsilon_{iab}\,\partial_{aL-1}(r^{-1}S_{bL-1}(t-r/c)), \qquad (2.32\,b)$$

$$h^{ij}_{\mathrm{can}\,1}[M] = -\frac{4}{c^4}\sum_{l\geqslant 2}\frac{(-)^l}{l!}\,\partial_{L-2}(r^{-1}\,{}^{(2)}M_{ijL-2}(t-r/c))$$
$$-\frac{8}{c^4}\sum_{l\geqslant 2}\frac{(-)^l\,l}{(l+1)!}\,\partial_{aL-2}(r^{-1}\epsilon_{ab(i}\,{}^{(1)}S_{j)\,bL-2}(t-r/c)), \qquad (2.32\,c)$$

and with, the indices α and β in (2.31) being raised with the flat metric $f^{\alpha\beta} = \mathrm{diag}\,(-1, +1, +1, +1)$,

$$w^0[W] = \sum_{l\geqslant 0}\partial_L(r^{-1}W_L(t-r/c)), \qquad (2.33\,a)$$

$$w^i[W] = \sum_{l\geqslant 0}\partial_{iL}(r^{-1}X_L(t-r/c))$$
$$+\sum_{l\geqslant 1}\{\partial_{L-1}(r^{-1}Y_{iL-1}(t-r/c))+\epsilon_{iab}\,\partial_{aL-1}(r^{-1}Z_{bL-1}(t-r/c))\}. \qquad (2.33\,b)$$

The suffix 'can' in $h^{\alpha\beta}_{\mathrm{can}\,1}[M]$ stands for 'canonical' because (2.32) is the canonical form of the linearized harmonic gothic perturbation used by Thorne (1980), which clearly differs from the general $h^{\alpha\beta}_1[M, W]$ only by an *infinitesimal* (harmonic in the domain D) *coordinate transformation* $\delta x^\alpha = Gw^\alpha[W]$. The coefficients in (2.30) have been chosen so that in the limiting case of a very slowly moving, negligibly self-gravitating, and negligibly self-stressed source (with mass density ρ) the 'new' M_L and S_L are simply given by (Thorne 1980)

$$M_{i_1 i_2 \ldots i_l} = \int \mathrm{d}^3x\,\rho x^{\langle i_1}x^{i_2}\ldots x^{i_l\rangle}, \qquad (2.34\,a)$$

$$S_{i_1 i_2 \ldots i_l} = \int \mathrm{d}^3x\,\rho\epsilon^{ab\langle i_1}x^{i_2}\ldots x^{i_l\rangle}x^a v^b. \qquad (2.34\,b)$$

In the general case M_L and S_L will not be related by such simple linear relations to the stress–energy tensor of the source but we shall still call them respectively the mass multipole moment (electric type) and the current multipole moment (magnetic type) of order l. In the following, these moments will only play the role of 'functional parameters' allowing to represent the general vacuum metric as a complicated nonlinear retarded functional of them. They will reacquire a direct physical meaning only in subsequent works studying the asymptotic behaviour at infinity of the metric or its matching to a (possibly strongly self-gravitating) source. Gathering our results we get finally the following representation theorem.

THEOREM 2.1. *In an open domain* $D = \{(\boldsymbol{x}, t)\,|\,r > r_0 \geqslant 0\}$ *the most general linearized harmonic vacuum metric,* $\mathfrak{g}^{\alpha\beta}_{\mathrm{lin}} = f^{\alpha\beta}+Gh^{\alpha\beta}_1$, *which admits a truncated multipolar expansion and which, when* $t \leqslant -T$, *is both stationary and asymptotically (in space) Minkowskian, i.e. the general solution of* (2.1)–(2.5), *admits the following representation*

$$\mathfrak{g}^{\alpha\beta}_{\mathrm{lin}} = f^{\alpha\beta}+Gh^{\alpha\beta}_1[M, W], \qquad (2.35)$$

with

$$h^{\alpha\beta}_1[M, W] = h^{\alpha\beta}_{\mathrm{can}\,1}[M]+\partial^\alpha w^\beta[W]+\partial^\beta w^\alpha[W]-f^{\alpha\beta}\,\partial_\mu w^\mu[W], \qquad (2.36)$$

Downloaded from rsta.royalsocietypublishing.org

where $M = \{M_L(u), S_L(u)\}$ *and* $W = \{W_L(u), X_L(u), Y_L(u), Z_L(u)\}$ *are* STF–*tensor functions of one real variable* u, *constant* (*i.e. independent of* u) *when* $u \leqslant -T$, *with* M, M_i *and* S_i *always constant, and where* $h^{\alpha\beta}_{\text{can}\,1}[M]$ *and* $w^{\alpha}[W]$ *are given by* (2.32) *and* (2.33).

This theorem is stated here for sufficiently differentiable h_1s, Ms and Ws (that is, of class C^n for some n). To find precisely the value of n appropriate to each M or W in order that $h_1^{\alpha\beta}$ be, for instance, $C^2(D)$, one should go back precisely through each step of the proof. Henceforth we shall assume, for simplicity's sake, that the Ms and Ws are all smooth ($C^{\infty}(\mathbb{R})$) and therefore also that $h_1^{\alpha\beta}$ is $C^{\infty}(D)$ for any $D = \{(\boldsymbol{x}, t)\,|\,r > r_0 \geqslant 0\}$ (the time-axis $r = 0$ being always excluded from D).

As a final comment let us indicate that the physical meaning, in the linearized theory, of the constancy of respectively M, M_i and S_i, is the conservation of, respectively, mass, centre of mass position and spin. Our constraint on past-stationarity means that we are always using a 'centre of mass frame' where the linear momentum is zero. It would probably be safe to strengthen our assumptions by requiring the choice of a suitable time-axis such that $M_i = 0$ ('mass centred frame') but it will be more convenient to leave M_i unconstrained.

3. Mathematical preliminaries

To investigate, within the assumptions (1.1)–(1.6), the existence and the structure of general solutions of the vacuum Einstein equations it will be necessary to use repeatedly the properties of special classes of functions of $\mathbb{R}^4$. We gather here the necessary definitions and some useful results. Some further mathematical tools, concerning special classes of functions of $\mathbb{R}^2$ and $\mathbb{R}^4$ will be expounded in §6 and §7.

3.1. *The* $O^N(r^N)$ *class*

Definition 3.1. A complex valued function of $\mathbb{R}^4$: $f(\boldsymbol{x}, t)$ is said to belong to the $O^N(r^N)$ class of functions, for some non-negative integer N (or, simply, is said to be $O^N(r^N)$) if the following properties hold: $\forall q \in \mathbb{N}$, $f^{(q)}(\boldsymbol{x}, t) := \partial^q f/\partial t^q$ exists everywhere in $\mathbb{R}^4$ and satisfies

(*a*) $\exists T \in \mathbb{R}$ such that $f^{(q)}(\boldsymbol{x}, t) = 0$ when $t \leqslant -T$;

(*b*) $f^{(q)}(\boldsymbol{x}, t)$ is of class $C^N(\mathbb{R}^4)$;

(*c*) $\forall t_0 \in \mathbb{R}$, $\exists M > 0$, $\exists d > 0$ such that (with $r := (\delta_{ij}\, x^i x^j)^{\frac{1}{2}}$)

$$(r < d) \Rightarrow (|f^{(q)}(\boldsymbol{x}, t_0)| < Mr^N). \tag{3.1}$$

In words, an $O^N(r^N)$ function is a past-zero function that is, together with all its time derivatives, both $C^N(\mathbb{R}^4)$ and $O(r^N)$ when $r \to 0$ with fixed t. By a slight abuse of notation we shall often write simply $f(\boldsymbol{x}, t) = O^N(r^N)$, it being understood that $f = O^N(r^N)$ and $g = O^N(r^N)$ do not imply $f = g$!

As a first consequence of the definition we can state that $\forall m \leqslant N$ the partial derivative $\partial_{i_1 \dots i_m} f^{(q)}(\boldsymbol{x}, t)$ ($i_1, \dots, i_m = 1, 2, 3$) is *uniformly*, over any time interval $[-T, t_0]$, $O(r^{N-m})$ when $r \to 0$, i.e. that $\forall m \leqslant N$, $\forall t_0 \in \mathbb{R}$, $\exists M > 0$, $\exists d > 0$ such that

$$(t \leqslant t_0 \text{ and } r < d) \Rightarrow \left(\left|\frac{\partial^m f^{(q)}}{\partial x^{i_1} \dots \partial x^{i_m}}(\boldsymbol{x}, t)\right| < Mr^{N-m}\right). \tag{3.2}$$

This is easily proven by applying to $\partial_{i_1 \dots i_m} f^{(q)}(\boldsymbol{x}, t)$ the Taylor formula (up to the order $N-m$) with integral remainder between the points $(\boldsymbol{x}, t)$ and $(\boldsymbol{0}, t)$. First, the case $m = 0$, together with

Downloaded from rsta.royalsocietypublishing.org

(3.1) shows that $\forall n \leqslant N-1$, $\partial_{i_1 \ldots i_n} f^{(q)}(\mathbf{0}, t) = 0$, then the continuity of $\partial_{i_1 \ldots i_N} f^{(q)}(\boldsymbol{x}, t)$ gives a uniform bound which leads to (3.2).

A useful criterion for proving that a function $f(\boldsymbol{x}, t)$ that is *a priori* defined and regular only *outside* the time-axis (where $r \neq 0$) can, however, be extended to an everywhere regular $O^N(r^N)$ function is the following.

LEMMA 3.1. *Let N and K be some non-negative integers and* $f(\boldsymbol{x}, t)$ *a function such that* $\forall q \in \mathbb{N}, f^{(q)}(\boldsymbol{x}, t)$ *is defined on* $\mathbb{R}^3_* \times \mathbb{R}$ (*with* $\mathbb{R}^3_* := \mathbb{R}^3 - \{\mathbf{0}\}$) *and that:*

(i) $\exists T$ *such that* $f^{(q)}(\boldsymbol{x}, t) = 0$ *when* $t \leqslant -T$,
(ii) $f^{(q)}(\boldsymbol{x}, t) \in C^N(\mathbb{R}^3_* \times \mathbb{R})$,
(iii) $\forall t_0 \in \mathbb{R}, \forall m \leqslant N, \exists M > 0, \exists d > 0, \exists p \geqslant 0$ *such that:*

$$(t \leqslant t_0 \text{ and } 0 < r < d) \Rightarrow (|\partial_{i_1 \ldots i_m} f^{(q)}(\boldsymbol{x}, t)| < M r^{K+1-m} |\lg r|^p), \tag{3.3}$$

then $f(\boldsymbol{x}, t)$ *can be extended, by continuity, to a function on* $\mathbb{R}^4$ *which is* $O^{N'}(r^{N'})$ *with* $N' = \inf(N, K)$.

An outline of the proof of this lemma is given in Appendix E.

The following basic stability properties of the $O^N(r^N)$ classes are easily deduced from the definition 3.1, equation (3.2), or lemma 3.1 (for simplicity's sake we use the 'notation' $O^N(r^N) + O^N(r^N) = O^N(r^N)$ instead of $f \in O^N(r^N)$ and $g \in O^N(r^N) \Rightarrow f+g \in O^N(r^N)$, etc. ...).

LEMMA 3.2. (*Algebraic and differential stability of* $O^N(r^N)$)*:*

(i) $O^N(r^N) + O^N(r^N) = O^N(r^N)$,
(ii) $O^N(r^N) \times O^N(r^N) = O^N(r^N)$,
(iii) $\forall q \in \mathbb{N}, \dfrac{\partial^q}{\partial t^q} O^N(r^N) = O^N(r^N)$,
(iv) $\forall m; 0 \leqslant m \leqslant N, \partial_{i_1 \ldots i_m} O^N(r^N) = O^{N-m}(r^{N-m})$,
(v) $\forall F(t) \in C^\infty(\mathbb{R}), \forall l \in \mathbb{N}, \forall p \geqslant 0$ (*with* $l+p > 0$), $\forall a \in \mathbb{Z}$ (*with* $n^L = n^{i_1} \ldots n^{i_l}$)*:*
- *if* $a \geqslant 1: F(t)\, n^L (\lg r)^p r^a O^N(r^N) = O^N(r^N)$,
- *if* $-(N-1) \leqslant a \leqslant 0: F(t)\, n^L (\lg r)^p r^a O^N(r^N) = O^{N+a-1}(r^{N+a-1})$.

The next important stability property of the $O^N(r^N)$ classes is the stability under the 'retarded integral', i.e. the integral operator of the 'retarded potential', defined (when it exists) as

$$(\Box_{\mathrm{R}}^{-1} f)(\boldsymbol{x}', t') := -\frac{1}{4\pi} \int \frac{\mathrm{d}^3 \boldsymbol{x}}{|\boldsymbol{x}' - \boldsymbol{x}|} f\left(\boldsymbol{x}, t' - \frac{1}{c}|\boldsymbol{x}' - \boldsymbol{x}|\right). \tag{3.4}$$

We shall often use the slightly improper notation: $(\Box_{\mathrm{R}}^{-1} f)(\boldsymbol{x}', t') = \Box_{\mathrm{R}}^{-1}(f(\boldsymbol{x}, t))$, distinguishing, when it is convenient, the 'field point' $(\boldsymbol{x}', t')$ where $\Box_{\mathrm{R}}^{-1} f$ is computed, and the 'source point' $(\boldsymbol{x}, t)$ on which one integrates.

LEMMA 3.3. (*Integral stability of* $O^N(r^N)$)*: if* $f(\boldsymbol{x}, t)$ *is* $O^N(r^N)$ *then there exist some functions* $F_{i_1 \ldots i_l}(t)$, *such that*

(i) $F_{i_1 \ldots i_l}(t) = 0$ *when* $t \leqslant -T$,
(ii) $F_{i_1 \ldots i_l}(t) \in C^\infty(\mathbb{R})$,
(iii) $(\Box_{\mathrm{R}}^{-1} f)(\boldsymbol{x}, t) - \sum_{l=0}^{N-1} x^{i_1} \ldots x^{i_l} F_{i_1 \ldots i_l}(t) \in O^N(r^N)$.

An outline of the proof is given in Appendix E. Symbolically we can write lemma 3.3 as ($L = i_1 \ldots i_l$, $n^L = n^{i_1} \ldots n^{i_l}$)

$$\Box_{\mathrm{R}}^{-1} O^N(r^N) = \sum_{l=0}^{N-1} n^L r^l F_L(t) + O^N(r^N). \tag{3.5}$$

Downloaded from rsta.royalsocietypublishing.org

3.2. *The L^n class*

The use of the $O^N(r^N)$ classes of functions is conveniently completed by the introduction of some other classes of functions which involve the nth power of the logarithm of r when $r \to 0$: the L^n class.

Definition 3.2. A complex valued function $f(\boldsymbol{x}, t)$ defined in $\mathbb{R}^3_* \times \mathbb{R}$ is said to belong to the L^n class of functions (for some $n \in \mathbb{N}$) if the following properties hold: for *any* positive integer N there exists a *finite* sum ($\hat{n}^Q = n^{\langle i_1} n^{i_2} \ldots n^{i_q \rangle}$ with $n^i = x^i/r$)

$$S_N(\boldsymbol{x}, t) = \sum_{p \leqslant n} F_{Qap}(t)\, \hat{n}^Q\, r^a\, (\lg r)^p, \tag{3.6a}$$

where $a \in \mathbb{Z}$, $p \in \mathbb{N}$ and $p \leqslant n$, and where the coefficients $F_{Qap}(t)$ are both $C^\infty(\mathbb{R})$ and zero in the past ($t \leqslant -T$ for some fixed T), such that the difference $f(\boldsymbol{x}, t) - S_N(\boldsymbol{x}, t)$, *a priori* defined only in $\mathbb{R}^3_* \times \mathbb{R}$, can be extended, by continuity, to be $O^N(r^N)$ (in $\mathbb{R}^4$),

$$\forall N \in \mathbb{N}, \quad f = S_N + O^N(r^N). \tag{3.6b}$$

Note that in definition 3.2 we have restricted the powers a of r that appear in S_N to be (positive or negative) integers because this is going to be the case in the following applications but it is not necessary to do so in general, the essential properties of the L^n classes are preserved if $a \in \mathbb{C}$.

In short, we can say that L^n is the class of functions that admit, when $r \to 0$, an asymptotic expansion to all order N, along the scale (or gauge) functions $r^a (\lg r)^p$, with $0 \leqslant p \leqslant n$, with coefficients admitting a finite multipolar expansion (the coefficients of which are smooth and zero in the past), and with a 'good' $O^N(r^N)$ remainder. Note that all $f \in L^n$ are $C^\infty(\mathbb{R}^3_* \times \mathbb{R})$ and that $n \leqslant m \Rightarrow L^n \subset L^m$.

The basic stability properties of the L^n classes under algebraic and differential operations are as follows.

Lemma 3.4. *If $f(\boldsymbol{x}, t) \in L^n$ and $g(\boldsymbol{x}, t) \in L^m$, then*

(i) $f(\boldsymbol{x}, t) + g(\boldsymbol{x}, t) \in L^{\sup(n,m)}$,
(ii) $f(\boldsymbol{x}, t) \,.\, g(\boldsymbol{x}, t) \in L^{n+m}$,
(iii) $\forall q \in \mathbb{N}, \forall p \in \mathbb{N}, \quad \partial_t^q\, \partial_{i_1 \ldots i_p} f(\boldsymbol{x}, t) \in L^n$.

Proof. (i) is easily deduced from definition 3.2 and lemma 3.2; to prove (ii) let us first notice that because of the uniqueness of asymptotic expansions for any given function f the powers of r must have a (possibly negative) minimum value $a_0(f)$, independent of N. Therefore, when considering the product $f.g$ one must, if $\inf(a_0(f), a_0(g)) \leqslant 0$, insert the 'expansions' of f and g ($f = S_{N'} + O^{N'}(r^{N'})$, ...) up to the order $N' = N + 1 - \inf(a_0(f), a_0(g))$, and then apply the last property of lemma 3.2; (iii) is also deduced from lemma 3.2 if one uses $N' = N + p$. ■

Finally the most important property of the L^n class is its behaviour under the integral operator $\Box_R^{-1}$ (the 'retarded integral' defined by (3.4)). The first difficulty is that the action of $\Box_R^{-1}$ on a function $f \in L^n$ is not *a priori* defined, because f will often (if $a_0(f)$ is too negative) not be locally integrable near $\boldsymbol{x} = 0$. Therefore a first step will consist in defining a convenient generalization of the operator $\Box_R^{-1}$ appropriate to the L^n class. There is no unique way to do that, but it proved very convenient to define such a generalization by using an approach based on complex analytic continuation. This type of approach to define an otherwise divergent integral has been introduced by Marcel Riesz (1938) and has since been used (and found indeed very useful and often superior to other methods) in many different contexts: quantum

Downloaded from rsta.royalsocietypublishing.org

field theory, classical theory of point particles, distribution theory, general relativistic dynamics of condensed bodies, to quote a few (for references and an introduction to the method see Damour 1983 *a*). In particular it was essentially the original method of Riesz which has been used recently, with a complex parameter denoted A, in a study of the general relativistic equations of motion, including radiation reaction effects, of two condensed objects (Damour 1983 *a*, *b*). Here we shall introduce a somewhat different approach (accordingly we shall use, instead of A, the letter B to denote the complex parameter). The final results of this approach are equivalent to the ones obtainable by means of Hadamard's concept of 'partie finie' (Hadamard 1932); however, the use of analytic continuation provides one with a more flexible and powerful technical tool.

Before coming to grips with the real problem, let us give a simple example of the use of complex analytic continuation to associate a finite number to a divergent integral. Let us consider a function, f, of one real variable r, of the following form: $f(r) = \sum c_a r^a$, where $\sum$ denotes a *finite* sum over $a \in \mathbb{Z}$ (so that $a \geqslant -n_0$ for some $n_0 \in \mathbb{N}$) and where the c_as are some (real or complex) coefficients. Because of the pole-like singular behaviour of $f(r)$ when $r \to 0$, the integral $I = \int_0^1 f(r)\, \mathrm{d}r$ will be in general undefined. Introducing a complex parameter B, let us consider the function $f_B(r) := r^B \times f(r)$. If B belongs to the right half-complex-plane $\mathbb{D} := \{B; \mathrm{Re}\,(B) > n_0 - 1\}$, $f_B(r)$ will be integrable so that we can define the following function of a complex variable: $B \in \mathbb{D} \to F(B) := \int_0^1 r^B f(r)\, \mathrm{d}r$. An easy explicit computation yields $F(B) = \sum c_a (B+a+1)^{-1}$, and $F(B)$ is seen to be analytic in its domain of definition $\mathbb{D}$. But now the function of B: $G(B) := \sum c_a (B+a+1)^{-1}$ is defined and analytic all over $\mathbb{C}' := \mathbb{C} - \mathbb{Z}$, the complex plane deprived of the integers. Hence we can extend, in a natural way, the definition of $F(B) = \int_0^1 r^B f(r)\, \mathrm{d}r$ to all Bs of $\mathbb{C}'$ as being the analytic function $G(B)$ ('analytic continuation' of $F(B)$). In this case we see immediately the possibility of this analytic continuation because we can use the same formula $\sum c_a (B+a+1)^{-1}$ all over $\mathbb{C}'$. In the general case, it will not be possible to exhibit such an explicit formula $(G(B))$ for $F(B)$ valid all over $\mathbb{C}'$, but the main interest of analytic continuation into some preassigned domain, which for the cases of concern to us here will be from $\mathbb{D}$ to $\mathbb{C}'$ (in fact $\mathbb{C}' \cup \mathbb{D}$), is that when it exists it is unique. Therefore it is sufficient to prove that a given function $F(B)$, originally defined and analytic only in some open domain of $\mathbb{C}$, can, by any procedure (for instance step by step), be continued, as an analytic function of B, all over $\mathbb{C}'$ to be able to speak of the uniquely defined $F(B)$ all over $\mathbb{C}'$. Coming back to our simple example $(F(B) = \int_0^1 r^B f(r)\, \mathrm{d}r)$, if $B = 0$ is not a singularity of the analytic continuation $G(B)$ of $F(B)$ we shall associate to the divergent integral $\int_0^1 f(r)\, \mathrm{d}r$ the number $I := G(0)$. On the other hand, if $B = 0$ is a singularity of $G(B)$ we shall associate to the divergent integral $\int_0^1 f(r)\, \mathrm{d}r$ the coefficient, say I, of the term in zeroth power of B in the Laurent expansion of $G(B)$ around $B = 0$. In both cases one has $I = \sum_{a \neq -1} c_a (a+1)^{-1}$ (and I coincides with Hadamard's 'partie finie' of $\int_0^1 f(r)\, \mathrm{d}r$). We shall now generalize this procedure to define a convenient generalization of the integral operator $\Box_{\mathrm{R}}^{-1}$ acting on an arbitrary function of the class L^n.

LEMMA 3.5. *Let* $f(\boldsymbol{x}, t) \in L^n$ *and* $B \in \mathbb{C}$, *then the function of* B, *calculated by* (3.4) *for any fixed point* $(\boldsymbol{x}', t') \in \mathbb{R}^3_* \times \mathbb{R}$,

$$F(B) = \Box_{\mathrm{R}}^{-1}((r/r_1)^B f(\boldsymbol{x}, t)) \tag{3.7}$$

has the following properties:

(i) $F(B)$ *is defined and is analytic in* B *in some half-plane* $\mathrm{Re}\,(B) > b_0$;

(ii) $F(B)$ *can be analytically continued all over* $\mathbb{C}' := \mathbb{C} - \mathbb{Z}$.

Downloaded from rsta.royalsocietypublishing.org

(In (3.7), r_1 is a constant, to be chosen at will, that plays no role in the following reasonings. Therefore we shall here choose units such that $r_1 = 1$. We will come back to the choice of r_1 in §5.)

Proof. By definition, we know that for any positive integer N, $r^B f$ can be written as a finite sum of terms of the type $F(t)\,\hat{n}^Q\, r^{B+a}\,(\lg r)^p$ (where, as remarked above, the powers of a are bounded from below, $\forall N: a \geqslant a_0$), plus a 'remainder' $r^B\, O^N(r^N)$. Moreover, each of the preceding terms are zero in the past. Therefore all possible problems concerning the convergence of $F(B)$, equation (3.7), for any fixed point $(\boldsymbol{x}', t') \in \mathbb{R}^3_* \times \mathbb{R}$, will come from the behaviour of $r^B f$ when $r \to 0$. First it is clear that if we choose $\mathrm{Re}\,(B)$ large enough $(> b_0 = -\inf(a_0, 0))$ all the terms constituting $r^B f$ will be continuous everywhere. Therefore $F(B)$ is at least defined in the half-plane $\mathrm{Re}\,(B) > b_0$. Now, if we formally differentiate $F(B)$ with respect to B under the integral sign, by using $(\partial/\partial B)\, r^B = r^B \lg r$, we are led to study the triple integral $\Box^{-1}_{\mathrm{R}}(r^B\,(\lg r) f)$. With B in the previous half-plane, this new integral has a compact support and its integrand is continuous in all its variables; therefore, by a standard theorem, $F(B)$ is analytic and $\partial F(B)/\partial B = \Box^{-1}_{\mathrm{R}}(r^B\,(\lg r) f)$. This proves (i). To prove (ii), we remark that $F(B)$ can be written as a finite sum of terms, plus a 'remainder', that are all separately defined and analytic in the half-plane $\mathrm{Re}\,(B) > b_0$. It is then sufficient to prove that each of these terms can be analytically continued, as far as wished, to the left in $\mathbb{C}'$. First, by iterating what has been just said about $\partial F(B)/\partial B$ we see that we can write

$$\Box^{-1}_{\mathrm{R}}(\hat{n}^Q\, r^{B+a}\,(\lg r)^p\, F(t)) = \partial^p/\partial B^p \{\Box^{-1}_{\mathrm{R}}(\hat{n}^Q\, r^{B+a}\, F(t))\}. \tag{3.8}$$

It is then sufficient to study the analytic continuation of terms without logarithms. Let us also *define*, as a short-hand notation, the following function of B (analytic in $\mathbb{C} - \{-a-q-3,\ -a+q-2\}$)

$$\Delta^{-1}(\hat{n}^Q\, r^{B+a}) := \frac{\hat{n}^Q\, r^{B+a+2}}{(B+a+2-q)\,(B+a+3+q)}. \tag{3.9}$$

This notation is justified by the easily verified fact that $(\Delta := \delta^{ij}\,\partial_{ij})$

$$\Delta(\Delta^{-1}(\hat{n}^Q\, r^{B+a})) = \hat{n}^Q\, r^{B+a}. \tag{3.10}$$

Using this notation let us now prove the identity (where $F(t)$ is any $C^\infty(\mathbb{R})$ function, zero in the past $(t \leqslant -T)$ and where ${}^{(2)}F(t) := \partial^2 F(t)/\partial t^2$):

$$\Box^{-1}_{\mathrm{R}}(F(t)\,\hat{n}^Q\, r^{B+a}) = F(t)\,\Delta^{-1}(\hat{n}^Q\, r^{B+a}) + \frac{1}{c^2}\Box^{-1}_{\mathrm{R}}({}^{(2)}F(t)\,\Delta^{-1}(\hat{n}^Q\, r^{B+a})). \tag{3.11}$$

The proof of (3.11) is as follows. If we first take $\mathrm{Re}\,(B)$ large enough, all the functions appearing in (3.11) will be well differentiable in $\mathbb{R}^4$, then it is easily seen that the equality deduced from (3.11) by applying the d'Alembert operator $\Box = \Delta - c^{-2}\,\partial_t^2$ to each side of (3.11) will be verified. Now, as both sides of (3.11) are well differentiable, are zero in the past and as their d'Alembertian are everywhere identical, we conclude from the uniqueness theorem for the wave equation (see e.g. Fock 1959, §92), that is essentially from the Kirchhoff formula, that the equation (3.11) must be true. More generally we obtain by iterating (3.11) (with $\Delta^{-k} := (\Delta^{-1})^k$ the kth iteration of the operator Δ^{-1} of (3.9))

$$\Box^{-1}_{\mathrm{R}}(F(t)\,\hat{n}^Q\, r^{B+a}) = \sum_{m=0}^{s} \frac{1}{c^{2m}}\, {}^{(2m)}F(t)\,\Delta^{-m-1}(\hat{n}^Q\, r^{B+a}) + \frac{1}{c^{2s+2}}\,\Box^{-1}_{\mathrm{R}}({}^{(2s+2)}F(t)\,\Delta^{-s-1}(\hat{n}^Q\, r^{B+a})). \tag{3.12}$$

Downloaded from rsta.royalsocietypublishing.org

The identities (3.11) and (3.12) have been proven only when $\mathrm{Re}(B)$ is large enough, but, by the uniqueness of analytic continuation, these identities will be valid in the whole domain of the B plane where any of the sides of these identities can be analytically continued. From (3.9) we see first that $\Delta^{-m-1}(\hat{n}^Q r^{B+a})$, with for definiteness $a \in \mathbb{Z}$, is certainly an analytic function of B in $\mathbb{C}' = \mathbb{C} - \mathbb{Z}$ (with some poles at some integer values of B) and that it involves r to the power $B+a+2+2m$. This last fact improves the convergence of the retarded integral appearing in the right-hand side of (3.12). Indeed, from the arguments used to prove (i), we see that this integral, and thus the right-hand side of (3.12) is analytic in the domain $\mathrm{Re}(B) > -a-2s-2$ except some integers. This proves, therefore, that the left-hand side of (3.12), i.e. $\Box_R^{-1}(F(t)\,\hat{n}^Q r^{B+a})$ can be analytically continued in the domain $\mathrm{Re}(B) > -a-2s-2$ except some integers. As this is true for any integer $s \geqslant 0$ (because $F(t) \in C^\infty(\mathbb{R})$) we conclude that $\Box_R^{-1}(F(t)\,\hat{n}^Q r^{B+a})$ can be analytically continued in $\mathbb{C}' = \mathbb{C} - \mathbb{Z}$. Thanks to (3.8), the same is true of $\Box_R^{-1}(F(t)\,\hat{n}^Q r^{B+a}(\lg r)^p)$. Finally, as the 'remainder' term of $F(B)$: $\Box_R^{-1}(r^B O^N(r^N))$ is clearly analytic for $\mathrm{Re}(B) > -N$ and that N can be chosen arbitrarily large, we conclude that $F(B)$ can indeed be analytically continued all over $\mathbb{C}'$. ∎

From the preceding proof we conclude also that $F(B)$, equation (3.7), will have at most multiple poles at some integer values of B (because the denominators in (3.9) and the differentiation $\partial^p/\partial B^p$ in (3.8) can generate at most such multiple poles). We are mainly interested in the neighbourhood of $B=0$ because we want to generalize the usual 'retarded integral' $\Box_R^{-1} f$, hence we introduce the following definition.

Definition 3.3. Given $f \in L^n$, we shall call 'finite part of the [generally divergent] retarded integral $\Box_R^{-1} f$' the constant term $C_0(\boldsymbol{x}', t')$ (zeroth power of B) in the Laurent expansion of the meromorphic function $\Box_R^{-1}[(r/r_1)^B f(\boldsymbol{x}, t)]$ near $B=0$:

$$\Box_R^{-1}[(r/r_1)^B f(\boldsymbol{x}, t)] = \sum_{j=-|j_0|}^{\infty} C_j(\boldsymbol{x}', t')\, B^j. \tag{3.13}$$

We will denote it (remember that the 'field point' $(\boldsymbol{x}', t') \in \mathbb{R}^3_* \times \mathbb{R}$)

$$C_0(\boldsymbol{x}', t') =: \underset{B=0}{\mathrm{FP}}\, \Box_R^{-1}[(r/r_1)^B f(\boldsymbol{x}, t)], \tag{3.14a}$$

or, more simply, if there is no ambiguity:

$$C_0 =: \mathrm{FP}\, \Box_R^{-1} f. \tag{3.14b}$$

The two fundamental properties of the operator $\mathrm{FP}\,\Box_R^{-1}$ are as follows.

THEOREM 3.1. *We have:*

$$\text{(i)}\quad \forall f \in L^n,\ \Box(\mathrm{FP}\,\Box_R^{-1} f) = f; \tag{3.15}$$

$$\text{(ii)}\quad f \in L^n \Rightarrow \mathrm{FP}\,\Box_R^{-1} f \in L^{n+1}. \tag{3.16}$$

(*Note the increase by one unit of the superscript n*).

Proof. The property (i) is obtained by noting first that, for $B \in \mathbb{C}'$, (choosing $r_1 = 1$)

$$\Box(\Box_R^{-1}(r^B f)) = r^B f. \tag{3.17}$$

(Indeed (3.17) is true if $\mathrm{Re}(B)$ is large enough for the function $r^B f$ to be sufficiently differentiable all over $\mathbb{R}^4$, then it is still true (in $\mathbb{R}^3_* \times \mathbb{R}$) for $B \in \mathbb{C}'$ by analytic continuation.) By using (3.13) and $r^B f = \mathrm{e}^{B \lg r} f = \sum_{j=0}^{\infty} B^j (\lg r)^j f/j!$ we see generally that

$$j < 0 \Rightarrow \Box C_j(\boldsymbol{x}, t) = 0, \tag{3.18}$$

$$j \geqslant 0 \Rightarrow \Box C_j(\boldsymbol{x}, t) = \frac{(\lg r)^j}{j!} f(\boldsymbol{x}, t). \tag{3.19}$$

Downloaded from rsta.royalsocietypublishing.org

The particular case $j = 0$ yields (3.15). To prove (ii), we have to control better the pole structure of $\Box_R^{-1}(r^B f)$ near $B = 0$. From the proof of lemma (3.5) we see that the $B = 0$ poles in

$$\Box_R^{-1}(r^B f) = \sum_{p \leqslant n} \frac{\partial^p}{\partial B^p}(\Box_R^{-1}(F(t)\,\hat{n}^Q r^{B+a})) + \Box_R^{-1}(r^B O^N(r^N)) \tag{3.20}$$

will come, if they exist, only from the 'S_N terms' in the right-hand side of (3.20). Moreover, we note that the poles appearing explicitly in (3.9) will always stay *simple* in the iterated operator $\Delta^{-m-1}(\hat{n}^Q r^{B+a})$ (because their positions differ by an odd integer and they jump, at each iteration, by 2 units). Hence we can write

$$\Delta^{-m-1}(\hat{n}^Q r^{B+a}) = (D(B)/B)\,\hat{n}^Q r^{B+a+2m+2}, \tag{3.21}$$

where $D(B)$ is a rational function of B which is analytic (no poles) at $B = 0$. Now the crucial step is to notice that the finite part at $B = 0$ of $\partial^p/\partial B^p(D(B)\,r^B/B)$ is proportional to the coefficient of B^{p+1} in the MacLaurin expansion of $D(B)\,r^B = D(B)\,e^{B \lg r}$, which is clearly a polynomial in $(\lg r)$ of order $p+1$, that is:

$$\underset{B=0}{\mathrm{FP}}\frac{\partial^p}{\partial B^p}(F(t)\,\Delta^{-m-1}(\hat{n}^Q r^{B+a})) = F(t)\,\hat{n}^Q r^{a+2m+2}\sum_{i=0}^{p+1} a_i\,(\lg r)^i. \tag{3.22}$$

Replacing now (3.12) into (3.20), and taking the finite part, we obtain

$$\underset{B=0}{\mathrm{FP}}\,\Box_R^{-1}(r^B f) = \sum_{p \leqslant n}\sum_{m=0}^{s}\underset{B=0}{\mathrm{FP}}\frac{\partial^p}{\partial B^p}\left(\frac{{}^{(2m)}F(t)}{c^{2m}}\Delta^{-m-1}(\hat{n}^Q r^{B+a})\right)$$
$$+\sum_{p \leqslant n}\underset{B=0}{\mathrm{FP}}\,\Box_R^{-1}\left\{\frac{\partial^p}{\partial B^p}\left(\frac{{}^{(2s+2)}F(t)}{c^{2s+2}}\Delta^{-s-1}(\hat{n}^Q r^{B+a})\right)\right\} + \Box_R^{-1}(O^N(r^N)). \tag{3.23}$$

The first sum in (3.23) is known from (3.22). Moreover, if s is large enough it can be checked that it is possible to commute FP and $\Box_R^{-1}$ in the second sum. Then, if $2s \geqslant N - a_0 - 1$, we see, thanks to (3.22) and lemma 3.2 (v), that the second sum is of the type $\Box_R^{-1}(O^N(r^N))$. This leads, for *any* given N, to

$$\underset{B=0}{\mathrm{FP}}\,\Box_R^{-1}(r^B f) = \sum_{p \leqslant n+1} F(t)\,\hat{n}^Q r^a\,(\lg r)^p + \Box_R^{-1}(O^N(r^N)), \tag{3.24}$$

where now the maximum value of the powers of $\lg r$ is $n+1$. A final recourse to lemma 3.3 (equation (3.5)) allows us to break the 'remainder term' in (3.24) in a 'sum term' plus a 'good' $O^N(r^N)$ remainder. This concludes the proof of theorem 3.1. ■

As a final comment we can say that theorem 3.1 proves that the operator FP $\Box_R^{-1}$ is a convenient generalization of the usual 'retarded integral operator' $\Box_R^{-1}$ when dealing with 'singular sources' f belonging to the L^n class. Indeed, it provides a solution g of the inhomogeneous wave equation $\Box g = f$ (which is, like f, zero in the past), and this solution lives in the next class L^{n+1}, so that it is possible to *iterate* the operator FP $\Box_R^{-1}$. This is what we are going to do in the next section.

4. General past-stationary MPM solution of the vacuum equations

The aim of this section is to construct algorithmically the most general (formal) solution of the vacuum Einstein equations fulfilling the assumptions (1.1)–(1.6). Let us recall first that by inserting the post-Minkowskian expansion (1.1)

$$\mathfrak{g}^{\alpha\beta}(x^\mu) := \sqrt{g}\,g^{\alpha\beta} = f^{\alpha\beta} + G h_1^{\alpha\beta} + \dots + G^n h_n^{\alpha\beta} + \dots \tag{4.1}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Downloaded from rsta.royalsocietypublishing.org

into the Einstein tensor density $2g(R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta}) =: 2g\,E^{\alpha\beta}$, one obtains

$$2gE^{\alpha\beta} = \sum_{n=1}^{\infty} G^n\{\partial_{\mu\nu} H^{\alpha\mu\beta\nu}(h_n) - N_n^{\alpha\beta}(h_m; m < n)\}, \tag{4.2}$$

where

$$H^{\alpha\mu\beta\nu}(h_n) := f^{\mu\nu}h_n^{\alpha\beta} + f^{\alpha\beta}h_n^{\mu\nu} - f^{\alpha\nu}h_n^{\beta\mu} - f^{\beta\mu}h_n^{\alpha\nu} \tag{4.3}$$

is linear in $h_n^{\alpha\beta}$, but where $N_n^{\alpha\beta}$ is a nonlinear algebraic function of the 'previous' h_m $(m < n)$ and their first and second partial derivatives (with $N_1^{\alpha\beta} \equiv 0$). Symbolically, the structure of $N_n^{\alpha\beta}$ is (with indices and coefficients suppressed)

$$N_n = \sum_{a=2}^{n} \sum_{\substack{m_1+\ldots+m_a=n \\ m_b<n}} \text{``}\partial\partial\text{''}\, h_{m_1}\ldots h_{m_a}, \tag{4.4}$$

where the two partial derivatives have to be distributed (possibly with repetition) among the h_{m_b}s (for instance $N_2 \sim h_1\,\partial^2 h_1 + \partial h_1\,\partial h_1$). From the contracted Bianchi identities ($E^{\alpha\beta}{}_{;\beta} \equiv 0$) and the structure of $H^{\alpha\mu\beta\nu}$, it is clearly seen that if the vacuum Einstein equations are satisfied up to order $n-1$ (included) then the 'nonlinear source' N_n satisfies identically

$$\partial_\beta N_n^{\alpha\beta} \equiv 0. \tag{4.5}$$

Imposing now the harmonicity condition (1.4) (at each order n) as well as the extra multipolar, past-stationarity and asymptotic conditions (1.2), (1.5) and (1.6), we get the following sequence of systems to be solved (with $\Box = f^{\mu\nu}\,\partial_{\mu\nu}$).

$$\Box h_n^{\alpha\beta} = N_n^{\alpha\beta}(h_m; m<n), \tag{4.6a}$$

$$\partial_\beta h_n^{\alpha\beta} = 0, \tag{4.6b}$$

$$h_n^{\alpha\beta} = \sum_{l=0}^{l_{\max}} h_{nL}^{\alpha\beta}(r,t)\,\hat{n}^L, \tag{4.6c}$$

$$t \leqslant -T \Rightarrow \partial_t h_n^{\alpha\beta} = 0, \tag{4.6d}$$

$$t \leqslant -T \Rightarrow \lim_{r\to\infty} h_n^{\alpha\beta} = 0. \tag{4.6e}$$

To find the most general solution of (4.6), i.e. the most general past-stationary–asymptotically Minkowskian vacuum MPM metric, in harmonic coordinates, we shall proceed in three steps. First (§4.1) to construct a 'particular' solution of (4.6), second (§4.2) to show that this 'particular' solution 'contains' the general solution, and third (§4.3) to show the use of a simpler 'canonical' solution.

4.1. *Construction of a particular solution:* $\mathscr{g}^{\alpha\beta}_{\text{part}}$

THEOREM 4.1. *Given a finite set of $C^\infty(\mathbb{R})$ STF tensors $M(u) = \{M_L(u), S_L(u)\}$ and $W(u) = \{W_L(u), X_L(u), Y_L(u), Z_L(u)\}$ $(l \leqslant l_{\max}[M,W])$ arbitrary except for the constraints that all the functions $M_L(u), \ldots, Z_L(u)$ are constant when $u \leqslant -T$ and that M, M_i and S_i are always constant, then there exists an algorithm which constructs, for any $n \in \mathbb{N}$, ten functions of $\mathbb{R}^3_* \times \mathbb{R}$ (which are functionals of M and W): $h^{\alpha\beta}_{\text{part}\,n}[M,W](\boldsymbol{x},t)$ solving (4.6) and such that $(h^{\alpha\beta}_{\text{part}\,n}(\boldsymbol{x},t) - h^{\alpha\beta}_{\text{part}\,n}(\boldsymbol{x},-T)) \in L^{n-1}$.*

Downloaded from rsta.royalsocietypublishing.org

Proof. Let us first decompose, for convenience, $M(u)$ and $W(u)$ in their 'stationary' parts ${}_{\mathrm{S}}M$, ${}_{\mathrm{S}}W$ (defined as their constant values before $-T$) and their 'dynamic' parts ${}_{\mathrm{D}}M(u) := M(u) - {}_{\mathrm{S}}M$, ${}_{\mathrm{D}}W(u) := W(u) - {}_{\mathrm{S}}W$ (which are *zero* before $-T$). Then the first step of the algorithm consists in defining $h^{\alpha\beta}_{\mathrm{part}\,1}[M, W]$ to be the right-hand side of (2.31), with (2.32) and (2.33), i.e.

$$h^{\alpha\beta}_{\mathrm{part}\,1}[M, W] := h^{\alpha\beta}_{\mathrm{can}\,1}[M] + \partial^{\alpha} w^{\beta}[W] + \partial^{\beta} w^{\alpha}[W] - f^{\alpha\beta}\,\partial_{\mu} w^{\mu}[W]. \tag{4.7}$$

We can clearly decompose $h_{\mathrm{part}\,1}[M, W]$ in a 'stationary' part ${}_{\mathrm{S}}h_{\mathrm{part}\,1} := h_{\mathrm{part}\,1}(\boldsymbol{x}, -T) = h_{\mathrm{part}\,1}[{}_{\mathrm{S}}M, {}_{\mathrm{S}}W]$ and a 'dynamic' part ${}_{\mathrm{D}}h_{\mathrm{part}\,1} := h_{\mathrm{part}\,1}(\boldsymbol{x}, t) - h_{\mathrm{part}\,1}(\boldsymbol{x}, -T) = h_{\mathrm{part}\,1}[{}_{\mathrm{D}}M, {}_{\mathrm{D}}W]$ (which is zero before $-T$). Theorem 2.1 states that $h_{\mathrm{part}\,1}$ solves (4.6) for $n = 1$, thus the only thing to prove is that ${}_{\mathrm{D}}h_{\mathrm{part}\,1} \in L^0$, that is, that if $F \in C^{\infty}(\mathbb{R})$ is zero in the past then $\hat{\partial}_L(r^{-1}F(t-r/c)) \in L^0$. This is proven by first applying Taylor's formula (with integral remainder) to $F(t-r/c)$ about $r = 0$ up to the order $N+l+1$. Then, expanding $\hat{\partial}_L$ by means of (A 31) leads to a sum of terms of the type $F^{(q)}(t)\,\hat{n}^L r^a$ plus a 'remainder' of the type $\sum_{0 \leqslant i \leqslant l} \hat{n}^L r^{N+1+i} \int_0^1 \mathrm{d}x\,(p!)^{-1}(1-x)^p F^{(q)}(t-rx/c)$. Now one checks that this remainder satisfies the hypotheses of lemma 3.1 with $K = N$. Therefore the remainder is $O^N(r^N)$ as was to be proven. Let us now assume, as an induction hypothesis, that we have already constructed all the $h^{\alpha\beta}_{\mathrm{part}\,m}[M, W]$ for $m \leqslant n-1$, satisfying theorem 4.1 and decomposed in stationary and dynamic (past-zero) parts, ${}_{\mathrm{S}}h_{\mathrm{part}\,m}$ and ${}_{\mathrm{D}}h_{\mathrm{part}\,m} \in L^{m-1}$. Replacing these h_ms in $N^{\alpha\beta}_n$ leads to a similar decomposition of N_n in a stationary ${}_{\mathrm{S}}N_n$ and a dynamic ${}_{\mathrm{D}}N_n$ (past-zero) part. We deal with ${}_{\mathrm{S}}N_n$ in Appendix C; let us here concentrate upon ${}_{\mathrm{D}}N_n$. Thanks to equation (4.4), to lemma 3.4 and to the structure of ${}_{\mathrm{S}}h_n$ discussed in Appendix C we see that ${}_{\mathrm{D}}N_n \in L^p$, where $p = \sup\,(\sum_{b=1}^{a}(m_b - 1))$ (with $\sum_{b=1}^{a} m_b = n$, $m_b < n$ and $2 \leqslant a \leqslant n$). The maximum is reached for $a = 2$ and is $p = n-2$. Hence the effective nonlinear 'source' ${}_{\mathrm{D}}N^{\alpha\beta}_n$ (dynamic part of the right-hand side of (4.6*a*)) belongs to L^{n-2}. Therefore if we solve (4.6*a*) by means of the operator $\mathrm{FP}\,\Box^{-1}_{\mathrm{R}}$, i.e. if we pose for $(\boldsymbol{x}', t') \in \mathbb{R}^3_* \times \mathbb{R}$

$${}_{\mathrm{D}}p^{\alpha\beta}_n(\boldsymbol{x}', t') := \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\mathrm{R}}\,((r/r_1)^B\,{}_{\mathrm{D}}N^{\alpha\beta}_n(\boldsymbol{x}, t)), \tag{4.8}$$

then by theorem 3.1, ${}_{\mathrm{D}}p^{\alpha\beta}_n \in L^{n-1}$ and

$$\Box\,{}_{\mathrm{D}}p^{\alpha\beta}_n = {}_{\mathrm{D}}N^{\alpha\beta}_n. \tag{4.9}$$

We still need to satisfy the harmonicity condition (4.6*b*). The 'divergence' $\partial_{\beta}\,{}_{\mathrm{D}}p^{\alpha\beta}_n$ is obtained by first computing the divergence of the right-hand side of (4.8) (without the FP sign), which thanks to the Bianchi identity (4.5) is equal to $B\,\Box^{-1}_{\mathrm{R}}\,((r/r_1)^B\,r^{-1}n^i\,{}_{\mathrm{D}}N^{\alpha i}_n)$ $(i = 1, 2, 3)$. Taking the finite part of the latter expression means finding the residue, at $B = 0$, of the same expression without the factor B in front. Hence

$$\partial_{\beta}\,{}_{\mathrm{D}}p^{\alpha\beta}_n = \underset{B=0}{\mathrm{Residue}}\,\Box^{-1}_{\mathrm{R}}((r/r_1)^B\,r^{-1}\,n^i\,{}_{\mathrm{D}}N^{\alpha i}_n). \tag{4.10}$$

As $r^{-1}\,n^i\,{}_{\mathrm{D}}N^{\alpha i}_n \in L^{n-2}$, it can be decomposed as a sum of terms of the type $\hat{n}^L r^a (\lg r)^p F(t)$ plus a remainder $O^N(r^N)$. Now the remainder, when multiplied by r^B, will not generate any pole at $B = 0$. We have seen in the proof of theorem 3.1 that the poles of $\Box^{-1}_{\mathrm{R}}(\hat{n}^L r^{B+a} F(t))$ were always simple ($\sim 1/B$). Therefore by formula (3.8) the poles of $\Box^{-1}_{\mathrm{R}}(\hat{n}^L r^{B+a} (\lg r)^p F(t))$ are

Downloaded from rsta.royalsocietypublishing.org

multiple ($\sim 1/B^{p+1}$) and we see that the residue of $\Box_R^{-1}(\hat{n}^L r^{B+a}(\lg r)^p F(t))$ is zero except in the logarithmic-free case ($p=0$). Now we have

$$\Box_R^{-1}(\hat{n}^L r^{B+a} F(t)) = -\frac{1}{4\pi}\int \mathrm{d}^3\boldsymbol{x}\,\hat{n}^L r^{B+a}\frac{F(t'-|\boldsymbol{x}'-\boldsymbol{x}|/c)}{|\boldsymbol{x}'-\boldsymbol{x}|}. \tag{4.11}$$

The poles of the right-hand side of (4.11) will come only from the integration on an arbitrary small neighbourhood of $\boldsymbol{x}=\boldsymbol{0}$ ($|\boldsymbol{x}|\leqslant\epsilon$). Expanding then $F(t'-|\boldsymbol{x}'-\boldsymbol{x}|/c)/|\boldsymbol{x}'-\boldsymbol{x}|$ in Taylor series around $\boldsymbol{x}=\boldsymbol{0}$ leads to a series of terms of the type $\partial'_Q(r'^{-1}F(t'-r'/c))\cdot(\int \mathrm{d}\Omega\hat{n}^L n^Q)\cdot(\int_0^\epsilon \mathrm{d}r r^{B+2+a+q})$. The angular integral is zero except if $q=l+2k$ ($k\in\mathbb{N}$) (see Appendix A), and the radial integral has a residue if and only if $a+q=-3$. When both conditions are met, the residue is proportional to $\hat{\partial}'_L(r'^{-1}\,{}^{(2k)}F(t'-r'/c))$ (it satisfies (3.18) as proven above). Changing the names of the space–time variables: $(\boldsymbol{x}',t')\to(\boldsymbol{x},t)$, we conclude that $\partial_\beta\,{}_{\mathrm{D}}p_n^{\alpha\beta}(\boldsymbol{x},t)$ is a finite sum of terms of the type $\hat{\partial}_L(r^{-1}F(t-r/c))$, where $F(t)$ is C^∞ and zero in the past. From §2 and Appendix A it can be uniquely decomposed by means of STF tensors. Hence we can write, in a *unique* manner,

$$\partial_\beta\,{}_{\mathrm{D}}p_n^{0\beta} = \sum_{l\geqslant 0}\partial_L(r^{-1}A_L(t-r/c)), \tag{4.12a}$$

$$\partial_\beta\,{}_{\mathrm{D}}p_n^{i\beta} = \sum_{l\geqslant 0}\partial_{iL}(r^{-1}B_L(t-r/c)) + \sum_{l\geqslant 1}\{\partial_{L-1}(r^{-1}C_{iL-1}(t-r/c)) + \epsilon_{iab}\,\partial_{aL-1}(r^{-1}D_{bL-1}(t-r/c))\}, \tag{4.12b}$$

where $A_L(u)$, $B_L(u)$, $C_L(u)$ and $D_L(u)$ are STF tensors which are $C^\infty(\mathbb{R})$ and zero when $u\leqslant -T$. Because the As, Bs, Cs and Ds are uniquely determined we can now algorithmically define a new object ${}_{\mathrm{D}}q_n^{\alpha\beta}$ by the formulae:

$${}_{\mathrm{D}}q_n^{00} := -cr^{-1}\,{}^{(-1)}A - c\,\partial_a(r^{-1}\,{}^{(-1)}A_a) + c^2\,\partial_a(r^{-1}\,{}^{(-2)}C_a), \tag{4.13a}$$

$${}_{\mathrm{D}}q_n^{0i} := -cr^{-1}\,{}^{(-1)}C_i - c\epsilon_{iab}\,\partial_a(r^{-1}\,{}^{(-1)}D_b) - \sum_{l\geqslant 2}\partial_{L-1}(r^{-1}A_{iL-1}), \tag{4.13b}$$

$$\begin{aligned}{}_{\mathrm{D}}q_n^{ij} := {}& -\delta_{ij}[r^{-1}B + \partial_a(r^{-1}B_a)] \\ & + \sum_{l\geqslant 2}\{(1/c)\,\partial_{L-2}(r^{-1}\,{}^{(1)}A_{ijL-2}) + 2\delta_{ij}\,\partial_L(r^{-1}B_L) - 6\partial_{L-1(i}(r^{-1}B_{j)L-1}) \\ & + \frac{3}{c^2}\partial_{L-2}(r^{-1}\,{}^{(2)}B_{ijL-2}) - \partial_{L-2}(r^{-1}C_{ijL-2}) - 2\partial_{aL-2}(\epsilon_{ab(i}r^{-1}D_{j)bL-2})\},\end{aligned} \tag{4.13c}$$

where

$${}^{(-1)}A(u) := \int_{-\infty}^{u}\mathrm{d}x\,A(x),\quad {}^{(-2)}A(u) := \int_{-\infty}^{u}\mathrm{d}x\,{}^{(-1)}A(x),$$

${}^{(1)}A(u) := \mathrm{d}A(u)/\mathrm{d}u$, ... and all the As, ..., Ds are taken at the retarded time $u=t-r/c$. ${}_{\mathrm{D}}q_n^{\alpha\beta}$ has been constructed so as to satisfy (in $\mathbb{R}^3_*\times\mathbb{R}$)

$$\Box\,{}_{\mathrm{D}}q_n^{\alpha\beta} = 0, \tag{4.14a}$$

$$\partial_\beta\,{}_{\mathrm{D}}q_n^{\alpha\beta} = -\partial_\beta\,{}_{\mathrm{D}}p_n^{\alpha\beta}. \tag{4.14b}$$

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

Let us note that it would be possible to construct, from the same As, ..., Ds other objects satisfying also (4.14); for instance, such an object is ${}_{\mathrm{D}}q_n'^{\alpha\beta}$ with ${}_{\mathrm{D}}q_n'^{00} = {}_{\mathrm{D}}q_n^{00}$, ${}_{\mathrm{D}}q_n'^{0i} = {}_{\mathrm{D}}q_n^{0i}$ but:

$$ {}_{\mathrm{D}}q_n'^{ij} = {}_{\mathrm{D}}q_n^{ij} + \sum_{l \geqslant 2} \{-3\delta_{ij}\, \partial_L(r^{-1}B_L) - \frac{3}{c^2}\partial_{L-2}(r^{-1}\,{}^{(2)}B_{ijL-2}) + 6\partial_{L-1(i}(r^{-1}B_{j)L-1})\}. \quad (4.15) $$

However, we adopt ${}_{\mathrm{D}}q_n^{\alpha\beta}$ rather than ${}_{\mathrm{D}}q_n'^{\alpha\beta}$ because the trace of ${}_{\mathrm{D}}q_n^{ij}$ is simpler: ${}_{\mathrm{D}}q_n^{ss} = -3[r^{-1}B + \partial_a(r^{-1}B_a)]$. If we finally define

$$ h^{\alpha\beta}_{\mathrm{part}\,n} := {}_{\mathrm{D}}p_n^{\alpha\beta} + {}_{\mathrm{D}}q_n^{\alpha\beta} + {}_{\mathrm{S}}h^{\alpha\beta}_{\mathrm{part}\,n}, \quad (4.16) $$

where the stationary part ${}_{\mathrm{S}}h_{\mathrm{part\,n}}$ is given in Appendix C, then we check that by construction $h_{\mathrm{part}\,n}$ solves (4.6) and that moreover, as ${}_{\mathrm{D}}p_n \in L^{n-1}$ and ${}_{\mathrm{D}}q_n$ (which has a structure similar to $h_{\mathrm{part}\,1}$) belongs to L^0 then, by lemma 3.4, ${}_{\mathrm{D}}h_{\mathrm{part}\,n} \in L^{n-1}$. Therefore, the theorem 4.1 is proven by induction. ■

In other words, we have constructed a particular MPM metric

$$ \mathscr{g}^{\alpha\beta}_{\mathrm{part}}[M, W] := f^{\alpha\beta} + \sum_{n=1}^{\infty} G^n\, h^{\alpha\beta}_{\mathrm{part}\,n}, \quad (4.17) $$

formal solution of the vacuum equations and past-stationary and past-asymptotically Minkowskian. We shall now prove that this 'particular' solution 'contains', in fact, the general solution of the problem.

4.2. *Construction of the general solution*: $\mathscr{g}^{\alpha\beta}_{\mathrm{gen}}$

THEOREM 4.2. *The most general past-stationary and past-asymptotically Minkowskian vacuum* MPM *metric in harmonic coordinates, i.e. the general solution of* (4.6) (*with* (4.1)), *can be formally expressed by means of the 'particular' solution* (4.17) *as*

$$ \mathscr{g}^{\alpha\beta}_{\mathrm{gen}} = \mathscr{g}^{\alpha\beta}_{\mathrm{part}}\left[\sum_{n=1}^{\infty} G^{n-1}M_n, \sum_{n=1}^{\infty} G^{n-1}W_n\right], \quad (4.18) $$

where the M_ns and W_ns are arbitrary finite sets of STF *tensors satisfying the hypotheses of theorem* 4.1, *and where all the series in G must be expanded and rearranged according to the usual rules of formal power series.*

Proof. The proof is by induction, let us just show on the first two steps how it works. Theorem 2.1 and definition (4.7) guarantee that the theorem is true at order G. Hence there exist some M_1s and W_1s such that the general h_1 is $h_{\mathrm{gen}\,1} = h_{\mathrm{part}\,1}[M_1, W_1]$. Then $h_{\mathrm{gen}\,2}$ must satisfy

$$ \Box h^{\alpha\beta}_{\mathrm{gen}\,2} = N_2^{\alpha\beta}(h_{\mathrm{part}\,1}[M_1, W_1]), \quad (4.19a) $$

$$ \partial_\beta h^{\alpha\beta}_{\mathrm{gen}\,2} = 0. \quad (4.19b) $$

We know already one particular solution of (4.19), namely $h_{\mathrm{part}\,2}[M_1, W_1]$, therefore the general solution will differ from it only by the general solution of the associated homogeneous system: $\Box h = 0$, $\partial \cdot h = 0$ (plus the boundary conditions (4.6*d*, *e*)), which is nothing else than the linearized problem of which we know the general solution. Hence we see that there exist some M_2, W_2 such that

$$ h_{\mathrm{gen}\,2} = h_{\mathrm{part}\,2}[M_1, W_1] + h_{\mathrm{part}\,1}[M_2, W_2]. \quad (4.20) $$

Downloaded from rsta.royalsocietypublishing.org

It is then easy to check that (4.20) can be rewritten as

$$f + Gh_{\text{gen 1}} + G^2 h_{\text{gen 2}} = \mathscr{g}_{\text{part}}[M_1 + GM_2, W_1 + GW_2] + O(G^3). \tag{4.21}$$

The same reasoning is readily extended to any order G^n. ■

4.3. *Coordinate transformations and the 'canonical' solution: $\mathscr{g}^{\alpha\beta}_{\text{can}}$*

Definition 4.1. Given arbitrary STF tensors $M = \{M_L(u), S_L(u)\}$ alone (satisfying the hypotheses of theorem 4.1) we define the 'canonical' metric as

$$\mathscr{g}^{\alpha\beta}_{\text{can}}[M] := \mathscr{g}^{\alpha\beta}_{\text{part}}[M, 0], \tag{4.22}$$

that is by annulling all the Ws in the 'particular' metric constructed in the proof of theorem 4.1.

It would seem at first sight that $\mathscr{g}_{\text{can}}$ is a very special type of vacuum (MPM) metric. However, we are going to show that it is geometrically (or physically) as general as the most general vacuum harmonic metric, because it differs from it only through an arbitrary harmonic coordinate transformation and an arbitrary redefinition of the 'physical' multipole moments $M = \{M_L(u), S_L(u)\}$. To do this we need first to control the general transformations between two harmonic coordinate systems (valid in the domain outside the time-axis).

THEOREM 4.3. *Given a finite set of* STF *tensors $W'(u) = \{W'_L(u), X'_L(u), Y'_L(u), Z'_L(u)\}$ constant in the past but otherwise arbitrary, and a general harmonic vacuum metric (parametrized by M, W), there exists an algorithm which constructs a coordinate transformation $T_{w[W']}$ to a new harmonic coordinate system of the type*:

$$x'^{\mu} = x^{\mu} + G w^{\mu}_{\text{part 1}}[W'] + \dots + G^n w^{\mu}_{\text{part } n}[W'] + \dots \tag{4.23}$$

such that all the $w_{\text{part } n}$s are stationary in the past ($t \leqslant -T$), satisfy ($w^{\mu}_{\text{part } n}(\boldsymbol{x}, t) - w^{\mu}_{\text{part } n}(\boldsymbol{x}, -T)) \in L^{n-1}$, and where $w^{\mu}_{\text{part 1}}[\,]$ is the functional given in (2.33). A functional dependence on M and W is understood in $w^{\mu}_{\text{part } n}[W']$ ($n \geqslant 2$).

Proof. The condition for $T_{w[W']}$ to lead to another harmonic system is simply (as we start from a harmonic one)

$$0 = \partial_{\alpha}(\mathscr{g}^{\alpha\beta}\partial_{\beta}x'^{\mu}) = \mathscr{g}^{\alpha\beta}(x)\,\partial_{\alpha\beta}x'^{\mu}. \tag{4.24}$$

Looking for $x'^{\mu} = x^{\mu} + Gw^{\mu}_1 + \dots + G^n w^{\mu}_n + \dots$ we must solve:

$$\mathscr{g}^{\alpha\beta}(x)\,\partial_{\alpha\beta}\{w^{\mu}_1 + Gw^{\mu}_2 + \dots + G^{n-1}w^{\mu}_n + \dots\} = 0. \tag{4.25}$$

We know the general harmonic vacuum metric, which can be written as $f^{\alpha\beta} + Gh^{\alpha\beta}_{\text{part 1}}[M, W] + \dots$, where the Ms and Ws represent formal series $\sum G^{n-1}M_n$, $\sum G^{n-1}W_n$, that we shall not need to explicitly expand here. We then get a sequence of equations to be solved for the ws:

$$\Box w^{\mu}_1 = 0, \tag{4.26}$$

$$\vdots$$

$$\Box w^{\mu}_n = -\sum_{m=1}^{n-1} h^{\alpha\beta}_{\text{part } n-m}[M, W]\,\partial_{\alpha\beta}w^{\mu}_m. \tag{4.27}$$

We choose as particular solution of (4.26) the right-hand side of (2.33) written for W'_L, X'_L, Y'_L and Z'_L; this defines $w^{\mu}_{\text{part 1}}[W']$. Then we proceed by induction as for the preceding algorithm

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

for $\mathscr{g}_{\text{part}}$, except that now we have no differential constraints on w^μ_n (comparable to $\partial_\beta h^{\alpha\beta}_n = 0$). We separate the ws in stationary and dynamic parts. At each induction stage ($n \geqslant 2$) we define

$$_{\mathrm{D}}w^\mu_{\text{part}\,n}[W'] := \mathrm{FP}\,\Box_{\mathrm{R}}^{-1}\left(-\sum_{m=1}^{n-1} h^{\alpha\beta}_{\text{part}\,n-m}[M,W]\,\partial_{\alpha\beta}w^\mu_{\text{part}\,m}[W']\right)_{\mathrm{D}}, \tag{4.28}$$

(where $(\;)_{\mathrm{D}}$ denotes the dynamic part of $(\;)$). This definition is meaningful, because $_{\mathrm{D}}h_{n-m} \in L^{n-m-1}$ and (by induction hypothesis) $_{\mathrm{D}}w_{\text{part}\,m} \in L^{m-1}$ imply that the right-hand side of (4.27) belongs to L^{n-2}, hence $_{\mathrm{D}}w_{\text{part}\,n} \in L^{n-2+1} = L^{n-1}$ (by theorem 3.1) ($_{\mathrm{S}}w$, treated in Appendix C, section C2, does not create any problem). ■

After having defined such a particular coordinate transformation we have a result analogous to theorem 4.2.

THEOREM 4.4. *The most general 'finite multipolar', past-stationary and past-asymptotically vanishing (in space) coordinate transformation of the form:* $x'^\mu = x^\mu + Gw^\mu$, *with* $w^\mu = w^\mu_1 + \ldots + G^{n-1}w^\mu_n + \ldots$, *which leads from an arbitrary harmonic metric* $\mathscr{g}_{\text{part}}[M, W]$ *to another one can be formally expressed as:*

$$w^\mu_{\text{gen}} = w^\mu_{\text{part}}\left[\sum_{n=1}^{\infty} G^{n-1}W'_n\right]. \tag{4.29}$$

This theorem is proven by the same method as used in the proof of theorem 4.2. The reason why it works lies in the fact that the general 'finite multipolar' solution of the homogeneous equation (4.26) has been found in §2 as being of the form (2.24), and that the tools of appendix A show that such a general solution can always be (uniquely) written as (2.33) that is precisely as $w_{\text{part}\,1}$ [*some* W].

We are now ready to state the last result of this section, showing that the canonical MPM metric $\mathscr{g}_{\text{can}}[M]$ defined by (4.22) contains the same geometrical (or physical) information as the general MPM metric $\mathscr{g}_{\text{gen}}[M, W]$, because they differ at most by a coordinate transformation and a redefinition of the multipole moments.

THEOREM 4.5. *Given a general harmonic vacuum* MPM *metric* $\mathscr{g}_{\text{gen}}$, *there exist an harmonicity preserving coordinate transformation* T_w *and a finite set of 'multipole moments'* M *(themselves expressed as a formal series* $\sum_{n=1}^{\infty} G^{n-1}M_n$ *as in theorem* 4.2) *such that:*

$$T_w\,\mathscr{g}_{\text{gen}} = \mathscr{g}_{\text{can}}[M]. \tag{4.30}$$

Proof. By the preceding theorems it is sufficient to prove that an arbitrary $\mathscr{g}_{\text{part}}[M', W']$ can be transformed by some coordinate transformation into some $\mathscr{g}_{\text{can}}[M]$. The proof is by induction; it consists in constructing the looked-for coordinate transformation as a formal product

$$T_w = \ldots \circ T_{w_{\text{part}}[G^{n-1}W_n]} \circ \ldots \circ T_{w_{\text{part}}[GW_2]} \circ T_{w_{\text{part}}[W_1]}. \tag{4.31}$$

Thanks to theorem 2.1, the first step is achieved by choosing $W_1 = -W'_1$ (if $W' = \sum_{n=1}^{\infty} G^{n-1}W'_n$), which effectively transforms $\mathscr{g}_{\text{part}}[M', W']$ into another harmonic metric (i.e. some $\mathscr{g}_{\text{part}}[M'', W'']$) that differs from $\mathscr{g}_{\text{can}}[M'_1]$ (if $M' = \sum_{n=1}^{\infty} G^{n-1}M'_n$) only by terms of formal order G^2. Hence, $W'' = GW''_2 + \ldots$. Then we choose $W_2 = -W''_2$ and apply $w_{\text{part}}[GW_2]$ and so on. ■

It must be noticed that the final 'physical multipole moments' M parametrizing the canonical final metric will be obtained as

$$M = M' + G\mathscr{M}_1[M', W'] + \ldots + G^n\mathscr{M}_n[M', W'] + \ldots \tag{4.32}$$

where the $\mathscr{M}_n$ will be complicated nonlinear integro-differential functionals of the moments

Downloaded from rsta.royalsocietypublishing.org

M', W' parametrizing the general solution $\mathscr{g}_{\text{gen}}$. The same applies to the coordinate transformation putting $\mathscr{g}_{\text{gen}}$ into canonical form; it can be written as $w_{\text{part}}[\overline{W}]$ with

$$\overline{W} = -W' + G\mathscr{W}_1[M', W'] + \ldots + G^n\mathscr{W}_n[M', W'] + \ldots. \tag{4.33}$$

Theorem 4.5 justifies (within our formal framework) the assumption made by several authors, notably Thorne (1980), that the general radiative metric can be expressed as a functional of only two sets of STF tensorial functions of one real variable, some 'mass' multipole moments $M_L(u)$ ('electric-type') and some 'current' multipole moments $S_L(u)$ ('magnetic type') with the only restrictions that M (the 'total mass'), M_i (the 'centre of mass position') and S_i (the 'intrinsic total angular momentum' or 'spin') be constant (in the 'centre of mass frame'). To avoid misunderstandings let us make it clear that, at this stage, the time-varying M_Ls and S_Ls are only formal functional parameters allowing us to represent the general past-stationary and past-asymptotically Minkowskian MPM harmonic vacuum metric $\mathscr{g}_{\text{gen}}$. They will acquire a more direct physical meaning only at a later stage, when matching to a source or when studying the asymptotic behaviour. However, in the case where we restrict our general time-varying multipole moments to be always constant, or simply if we look at the structure of $\mathscr{g}_{\text{gen}}$ before the time $-T$, or for any time t but for large r (spatial infinity) we recover a physical situation well studied by many authors (notably, Geroch 1970; Hansen 1974; Thorne 1980; Beig & Simon 1981; Beig 1981; Gürsel 1983; Simon & Beig 1983 and references therein). In this situation, the M_Ls and the S_Ls have been shown to have a well-defined geometrical meaning.

5. Near-zone structure of the general solution

In the preceding section we have shown how to construct algorithmically the most general MPM *vacuum* metric in harmonic coordinates. Evidently, as we are mainly interested in studying the gravitational radiation emitted by an actual source, the previous vacuum metric can be of use only outside the source. For physical applications it is useful to distinguish several spatial regions outside the source. If a denotes the (characteristic) size of the source and λ the characteristic wavelength of any radiation field emitted by the source, one traditionally distinguishes: a 'near zone' ($r \ll \lambda$), an intermediate or 'transition zone' ($r \sim \lambda$) and a 'far zone' also called radiation or 'wave zone' ($r \gg \lambda$) (see e.g. Jackson 1975, p. 392). A far as its radiative properties are concerned, a field behaves quite differently in the preceding three regions (see e.g. Finn 1985). Moreover, when nonlinear effects are important (as it is the case for the gravitational field) one must also distinguish between strong-field regions and weak-field regions ($r \gg Gc^{-2}M$, M being the characteristic mass of the source). Thorne (1980) has also introduced a further distinction between a 'local wave zone' and a 'distant wave zone' that need not concern us now. One expects *a priori* the general MPM vacuum metrics (investigated by Bonnor & Rotenberg 1966; Thorne 1980; and this work) to be able to provide good approximations to the actual metric only in the weak-field part of the region outside the source (i.e. $r > a$ and $r \gg Gc^{-2}M$) (this is when assuming to deal with a very large number of multipoles; if one keeps only a few multipoles one must probably stay far away from the source: $r \gg a$). On the other hand, the radiative behaviour of the field, i.e. the distinction near/transition/far zones, does not seem to place any further limitations on the approximate validity of the MPM expansions. In §7 we shall investigate the far-zone behaviour of the general

Downloaded from rsta.royalsocietypublishing.org

MPM metric; here we shall concentrate on its behaviour in the near zone ($r \ll \lambda$) outside the source ($r > a$). This presupposes that $a \ll \lambda$, which defines the so-called 'slow sources' ($a \ll \lambda$ implies generally that the characteristic velocity within the source $v \ll c$). To formalize the asymptotic behaviour of $\mathscr{g}^{\alpha\beta}$ when $r \ll \lambda$ (and $a \ll \lambda$), it is convenient to introduce the characteristic period P of the waves emitted (so that $\lambda = cP$), to use units for space and time such that $a = O(1)$ and $P = O(1)$ (and thus $v = O(1)$), to use as constant r_1 in equation (4.8): $r_1 = \lambda = cP$, and then to consider a *sequence* of sources which become more and more 'slow' ($a/\lambda \rightarrow 0$), which means in our source-based system of units that the number c measuring the velocity of light goes to infinity: $c \rightarrow \infty$ (J. Ehlers, personal communication 1983).† We have seen ((2.32) and (2.34)) that if we attribute to $M_L(t)$ and $S_L(t)$ their usual physical dimensions (so that $M_L \sim Ma^l$, $S_L \sim Ma^{l+1}/P$ stay $O(M)$ when $c \rightarrow \infty$), then explicit powers of c appear in $h_1^{\alpha\beta}$. Let $M = \{M_L, S_L\}$ denote the finite set of the 'multipoles' allowing us to construct $\mathscr{g}_{\text{can}}[M]$. With the preceding choice of units, each $h_{\text{can}\,n}[M]$ becomes a function of $\boldsymbol{x}$, t and c and a multilinear functional (of order n) of the elements of M. In other words, $h_{\text{can}\,n}$ is a sum of terms each of which is, as concerns its algebraic structure, an arbitrary element (say E_n) of the nth tensorial power of M: M^n, i.e. a tensor product of n multipoles chosen among M:

$$E_n = M_{L_1} M_{L_2} \dots M_{L_{n-s}} S_{L_{n-s+1}} \dots S_{L_n}. \tag{5.1}$$

Each such E_n is then multiplied (with contractions) by some Levi–Civita and Kronecker symbols. To each $E_n \in M^n$ we can associate two integers: $s(E_n)$ the number of 'current moments' among the n multipoles, and $b(E_n)$ the total number of indices among the M_Ls and the $\epsilon_{ija} S_{aL-1}$s appearing in E_n (when endowing the S_Ls with their natural ϵ associates), i.e.

$$s(E_n) := s, \tag{5.2a}$$

$$b(E_n) := s + \sum_{a=1}^{n} l_a. \tag{5.2b}$$

With these notations it is easy to prove by induction (starting with (2.32) and using definitions (4.8), (4.13) and (4.16) together with the fact that $\Box_{\text{R}}^{-1} f(\boldsymbol{x}/c, t) = c^2 g(\boldsymbol{x}'/c, t')$) that the dependence on c of $h_{\text{can}\,n}(\boldsymbol{x}, t, c)$ can be reduced to

$$h_{\text{can}\,n}(\boldsymbol{x}, t, c) = \sum_{E_n \in M^n} \frac{1}{c^{3n+b(E_n)}} h_{E_n}(\boldsymbol{x}/c, t), \tag{5.3}$$

where h_{E_n}, which involves only the ratio $\boldsymbol{x}{:}c$, is algebraically constructed only from a single element E_n of M^n. Combining now the 'factorization result' (5.3) with the previously demonstrated fact (§4) that ${}_{\text{D}}h_n \in L^{n-1}$ and the known structure of ${}_{\text{S}}h_n$ (Appendix C) we find that, for any positive integer N, we can write:

$$h_{\text{can}\,n}(\boldsymbol{x}, t, c) = \sum_{E_n \in M^n} \frac{1}{c^{3n+b(E_n)}} \{\sum F_{Qap}(t)\, \hat{n}^Q (r/c)^a (\lg(r/c))^p + R_N(\boldsymbol{x}/c, t)\}, \tag{5.4}$$

where the functions $F_{Qap}(t)$ are C^∞ and constant (and even zero for $p \neq 0$ or $a \neq -(n+b(E_n)-s(E_n))$) when $t \leqslant -T$, where $-(n+b(E_n)-s(E_n)) \leqslant a \leqslant N$ (as shown by induction), where $0 \leqslant p \leqslant n-1$ and where $R_N(\boldsymbol{y}, t)$ is an $O^N(r^N)(\boldsymbol{y}, t)$ function.

† For a definition of a precise framework in which one can investigate the limit process $c \rightarrow \infty$ see Ehlers (1984) and references therein.

Downloaded from rsta.royalsocietypublishing.org

In the case of 'slow sources', with a convenient choice of units ($a = O(1)$, $P = O(1)$, $c \to \infty$) we can consider that for all fixed r, $r/c \sim r/cP = r/\lambda \to 0$, and therefore that the formula (5.4), which came originally from an asymptotic expansion when $r \to 0$ (mathematically), can be reread as an asymptotic expansion when $r/c \sim r/\lambda \to 0$ (with r fixed), i.e. as a 'near-zone expansion' giving the asymptotic behaviour of the metric outside a slow source when $a < r \ll \lambda$. As such, (5.4) confirms and generalizes (because 'tail terms' have been fully taken into account here) a result of Thorne (1980; §IX). Note however that the $F_{Qap}(t)$ of (5.4) are complicated nonlinear retarded functionals of the multipole moments and not, as in Thorne's incomplete treatment, antiderivatives of contracted products of the derivatives of the moments. An interesting by-product of (5.4) is obtained by re-expressing the preceding 'near-zone expansion' in a more formal way as an expansion when $c \to \infty$ with fixed r, that is, by using a common terminology, as a '*post-Newtonian expansion*'. Then, as a corollary of (5.4), we see that, up to an arbitrary order N, $h_{\mathrm{can}\, n}(c)$ admits the following post-Newtonian expansion:

$$h_{\mathrm{can}\, n}(c) = \sum_{E_n \in M^n} \frac{1}{c^{2n+s(E_n)}} \left\{ \sum_{\substack{0 \leqslant p \leqslant n-1 \\ 0 \leqslant k \leqslant N}} \frac{(\lg c)^p}{c^k} + O\left(\frac{1}{c^N}\right) \right\}. \tag{5.5}$$

The remarkable fact is that (5.5) proves that g_{can} admits a post-Newtonian expansion of arbitrary order only if one uses as scale (or gauge) functions the $(\lg c)^p/c^k$ ($p, k \in \mathbb{N}$) (or some finer set of gauge functions). This proves that the usual post-Newtonian assumptions, according to which g admits a post-Newtonian expansion along the simple powers $1/c^k$, is inconsistent with the nonlinear structure of general relativity. This inconsistency has, in fact, already shown up in the higher orders of post-Newtonian expansions where the assumption of simple powers $1/c^k$ leads to the appearance of divergent integrals (see e.g. Kerlick 1980; Futamase 1983). Anderson *et al.* (1982) have pointed out, using some matching arguments and a partial integration of Einstein's equations, the necessity to complete the set of simple powers $1/c^k$ ($k \in \mathbb{N}$) by $(\lg c)/c^k$ ($\epsilon^k \lg \epsilon$ in their notation) when considering the 'near-zone' expansion. Here we have shown directly that the integration of the full vacuum Einstein equations, up to arbitrary post-Minkowskian order, necessitates the extension of the set of scale functions to the $(\lg c)^p/c^k$ ($p, k \in \mathbb{N}$). It is notable that such simple scale functions are sufficient; one could have *a priori* expected higher order terms to necessitate the use of $c^{-k} \lg (\lg c)$ terms, for instance.

6. The retarded integral of a multipolar extended source

The mathematical tools introduced in §3, namely the $O^N(r^N)$ and the L^n classes of functions, have been useful both to construct the general MPM metric and to study its near-zone behaviour. However, in order to study the far-zone behaviour of the general MPM metric it is necessary to introduce some new mathematical tools: an explicit formula for the retarded integral of a multipolar extended source, a class of functions of *two* variables (r, $u = t - r/c$), the $O^\infty(1/r^N)$ class, and a class of functions of four variables, the $\mathscr{L}^n$ class. We shall discuss here the former integration formula, and we shall introduce the latter classes of functions in the next section directly devoted to the far-zone behaviour of the general MPM metric.

It is known (see, for example, Fock 1959) that, if an 'extended source' $S(\boldsymbol{x}, t)$ is sufficiently

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

regular (e.g. $C^2(\mathbb{R}^4)$) and is zero in the past, there exists one and only one past-zero $C^2(\mathbb{R}^4)$ solution of the inhomogeneous wave equation

$$\Box u(\boldsymbol{x}, t) = S(\boldsymbol{x}, t), \tag{6.1 a}$$

namely the retarded integral of S:

$$u(\boldsymbol{x}', t') = \Box_{\mathrm{R}}^{-1}(S(\boldsymbol{x}, t)) = -\frac{1}{4\pi}\int \frac{\mathrm{d}^3\boldsymbol{x}}{|\boldsymbol{x}'-\boldsymbol{x}|} S(\boldsymbol{x}, t'-|\boldsymbol{x}'-\boldsymbol{x}|/c). \tag{6.2}$$

In general, the explicit solution (6.2) is somewhat awkward to handle because it contains a triple integral. For instance it is not straightforward to use (6.2) to control the far-zone behaviour of u, knowing the far-zone behaviour of S. However, if S has a known (orbital) multipolarity l, that is if there exists a function of two variables, denoted also by the letter S, such that

$$S(\boldsymbol{x}, t) = \hat{n}^L S(r, t-r/c), \tag{6.1 b}$$

where, for the sake of convenience, we have suppressed indices and have used as variables the radius r and the retarded time $t-r/c$ instead of r and t, then the retarded integral (6.2) can be reduced (under the conditions of theorem 6.1 below) to a simpler line integral of some integrand constructed from an antiderivative of $f(r) = (2/r)^{l-1}S(r, s)$ (s fixed). Let us define the following functions:

$$R(r, s) := r^l \int_0^r \mathrm{d}x \frac{(r-x)^l}{l!} (2/x)^{l-1} S(x, s), \tag{6.3}$$

and (with $c = 1$):

$$u_L(\boldsymbol{x}, t) := \int_{-\infty}^{t-r} \mathrm{d}s\, \hat{\partial}_L \left\{\frac{R[\frac{1}{2}(t-r-s), s] - R[\frac{1}{2}(t+r-s), s]}{r}\right\}. \tag{6.4}$$

Then, we have the following theorem.

THEOREM 6.1. *Let $S(r, u)$ be a complex valued function on $\mathbb{R}^2_+ := \{(r, u) | r \geqslant 0\}$ that satisfies:*

(*a*) *there exists T such that $S(r, u) = 0$ when $u \leqslant -T$;*

(*b*) *there exists a positive integer N such that, for all $i \leqslant N+l+1$, $\partial^i/\partial r^i\, (S(r, u))$ exists and belongs to $C^N(\mathbb{R}^2_+)$;*

(*c*) *when $r \to 0$, $\partial^i/\partial r^i\, (S(r, u))$ is $O(r^{2N+l+1-i})$ uniformly in u, i.e. $\forall u_0$, $\forall i \leqslant 2N+l+1$, $\exists M > 0$, $\exists d > 0$ such that when $u \leqslant u_0$ and $r < d$, $|\partial^i/\partial r^i\, (S(r, u))| < M r^{2N+l+1-i}$.*

Then the function on $\mathbb{R}^4$ $u_L(\boldsymbol{x}, t)$ defined by (6.4) (with (6.3)) is $C^N(\mathbb{R}^4)$, and when $N \geqslant 2$, u_L is equal to the retarded integral of the 'source' (6.1 b), i.e.:

$$N \geqslant 2 \Rightarrow u_L(\boldsymbol{x}', t') = \Box_{\mathrm{R}}^{-1}(\hat{n}^L S(r, t-r)). \tag{6.5}$$

Proof. A formal way to find (6.5) uses the multipolar expansion of the retarded Green function $G_{\mathrm{R}} = \delta(t'-t-|\boldsymbol{x}'-\boldsymbol{x}|)/|\boldsymbol{x}'-\boldsymbol{x}|$ (see Appendix D). We now outline an indirect but rigorous proof of (6.5) under the hypotheses of theorem 6.1.

By using the lemma E 1 of Appendix E it is easy to deduce from the hypotheses of theorem 6.1, that, successively,

$$\forall i \leqslant N+l, \quad \hat{n}^L(\partial^i/\partial r^i)\, S(r, u) \in C^N(\mathbb{R}^4),$$

$$\forall i \leqslant N+1, \quad (\partial^i/\partial r^i)\, [(2/r)^{l-1}S(r, u)] \in C^N(\mathbb{R}^2_+),$$

$$\forall j \leqslant 2N+2l+2, \quad (\partial^j/\partial r^j)\, R(r, s) \text{ is uniformly } O(r^{2N+2l+3-j}) \text{ when } r \to 0,$$

$$\forall j \leqslant N+2l+2, \quad (\partial^j/\partial r^j)\, R(r, s) \in C^N(\mathbb{R}^2_+).$$

Downloaded from rsta.royalsocietypublishing.org

Then applying Taylor's formula, with integral remainder, at order $N+l+2$ to $R[\frac{1}{2}(t-s\pm r), s]$ (around $r=0$) leads one to express $\hat{\partial}_L\{(1/r)\,[R(\frac{1}{2}(t-s-r), s)-R(\frac{1}{2}(t-s+r), s)]\}$ as a sum of terms of the type $\hat{n}^L r^{2j-l}(\partial^{2j+1}R/\partial r^{2j+1})\,(\frac{1}{2}(t-s), s)$, with $l \leqslant j \leqslant (N+l+1)/2$ plus a remainder: $\hat{n}^L r^{N+1}P(r, t, s)$, where, thanks to the results above, P is checked to be $C^N(\mathbb{R}^3_+)$. Then $u_L(\boldsymbol{x}, t)$ ((6.4)) can be written as a sum of terms of the type $\hat{n}^L r^{2j-l}\int_{r/2}^{\infty}\mathrm{d}x(\partial^{2j+1}R/\partial r^{2j+1})\,(x, t-2x)$ plus a remainder:

$$\hat{n}^L r^{N+1}\int_{\frac{1}{2}r}^{\infty}\mathrm{d}x\,P(r, t, t-2x).$$

The results above, thanks to lemma E 1, show then that $u_L \in C^N(\mathbb{R}^4)$. Now let us define

$$f_L(\boldsymbol{x}, t, s) := \hat{\partial}_L\{r^{-1}[R(\tfrac{1}{2}(t-s-r), s)-R(\tfrac{1}{2}(t-s+r), s)]\}.$$

It is evident that $\Box f_L = 0$ if $r \neq 0$, i.e. in the domain $\Delta := \mathbb{R}^3_* \times \mathbb{R}$. Therefore the value of $\Box u_L$ comes only from the differentiation of the upper limit of the integral in (6.4). One finds that, in Δ (the replacement $s = t-r$ being done last)

$$\Box u_L(\boldsymbol{x}, t) = -2\,[(1/r)(\partial/\partial r+\partial/\partial t)\,(rf_L(\boldsymbol{x}, t, s))]_{s=t-r}.$$

Expanding f_L by means of (A 35*a*) and noting that, $\forall k \leqslant l-1$, $(\partial^k/\partial r^k)\,R(0, s) = 0$, leads to the following complicated expanded expression for $\Box u_L$:

$$\Box u_L = \left[l(l+1)\frac{\hat{n}^L}{(-2)^{l-1}}\sum_{k=0}^{l+1}\frac{(-)^k(2l-k)!}{k!(l+1-k)!}r^{k-l-2}\frac{\partial^k R}{\partial r^k}(r, s)\right]_{s=t-r},$$

which can be put in the simple form:

$$\Box u_L(\boldsymbol{x}, t) = \hat{n}^L(\tfrac{1}{2}r)^{l-1}\left[\frac{\partial^{l+1}}{\partial r^{l+1}}\left(\frac{R(r, s)}{r^l}\right)\right]_{s=t-r}.$$

Now by (6.3) we see that, for fixed s, $g(r) = R(r, s)/r^l$ is an anti-derivative of order $l+1$ of $f(r) = (2/r)^{l-1}\,S(r, s)$; therefore in the domain Δ (outside the time-axis) we have

$$\Box u_L(\boldsymbol{x}, t) = \hat{n}^L S(r, t-r). \tag{6.6}$$

When $N \geqslant 2$ both sides of the latter equation are at least continuous all over $\mathbb{R}^4$ (because u_L and $\hat{n}^L S$ are $C^N(\mathbb{R}^4)$), thus we see that the latter equality is also valid in $\mathbb{R}^4$ (i.e. including the time-axis). Finally it can be seen from (6.3) and (6.4) that $u_L(\boldsymbol{x}, t)$ is zero when $t-r \leqslant -T$, therefore by the uniqueness theorem for the past-zero solution of the wave equation (6.1) we conclude that u_L, as defined by (6.3) and (6.4) is indeed the retarded integral of $\hat{n}^L S(r, t-r)$. ■

Remark 1. By using (A 36), it is easily checked that (6.4) is still valid if we replace $r^{-l}R(r, s)$, defined by (6.3) as being *the* $(l+1)$th antiderivative (with respect to r) of $(2/r)^{l-1}\,S(r, s)$, which vanishes, together with its first l derivatives, in $r=0$, by *any other* $(l+1)$th antiderivative of $(2/r)^{l-1}S(r, s)$ (for fixed s). For instance, we can replace (6.3) by:

$$R_a(r, s) := r^l\int_a^r \mathrm{d}x\,\frac{(r-x)^l}{l!}\,(2/x)^{l-1}\,S(x, s), \tag{6.7}$$

i.e. by the $(l+1)$th antiderivative of $(2/r)^{l-1}S(r, s)$, which vanishes, together with its first l derivatives, in $r=a$, where a can be any (sufficiently regular) function of t and s. Two choices of a (besides $a=0$ used in (6.3)) can be of interest in the applications of (6.4): $a = (t-s)/2$

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

or $a=+\infty$.The choice $a=\frac{1}{2}(t-s)$ is useful when one is interested in checking the domain of dependence of u_L. Indeed it is easily checked that, when $a=\frac{1}{2}(t-s)$, all the integrations appearing in (6.4) and (6.7) are limited to a domain of the (r, t) half plane which is precisely the (r, t)-projection of the support of the retarded Green function, (the past light cone of $\boldsymbol{x}', t'$) i.e. $\{(r, t) \mid r \geqslant 0, t'-r' \leqslant t+r \leqslant t'+r' \text{ and } t-r \leqslant t'-r'\}$. This provides an explicit check of the causal nature of the solution (6.4). On the other hand the choice, when it is possible, $a=+\infty$ (the causal nature of which is not obvious although correct) is convenient when one is interested in relating the far-zone behaviour of u_L to the far-zone behaviour of S.

Remark 2. Whatever be the choice of the antiderivative, $R(r, s)$, together with its first $(l-1)$ derivatives, will be zero in $r=0$ (this fails if $l=0$, but then (6.8) below is clearly true). This implies that (6.4) can be rewritten as, e.g.,

$$u_L(\boldsymbol{x}, t)=\hat{\partial}_L\left\{\frac{1}{r}\int_{-\infty}^{t-r} \mathrm{d}s\, R_a[\tfrac{1}{2}(t-r-s), s]\right\}-\int_{-\infty}^{t-r} \mathrm{d}s\, \hat{\partial}_L\left\{\frac{R_a[\frac{1}{2}(t+r-s), s]}{r}\right\}, \tag{6.8}$$

because it is easily checked that the terms coming from the differentiation of the upper limit of the first integral are proportional to $(\partial^k/\partial r^k)\, R_a(0, s)$ for $k \leqslant l-1$. The first term in the right-hand side of (6.8) is clearly a solution of the homogeneous wave equation outside the time-axis, of the purely retarded type (§2). Therefore if one is only interested in finding one particular solution, outside the time-axis $(r \neq 0)$, of the inhomogeneous wave equation (6.6), which vanishes in the past with the source S, it would be sufficient to use only the second term in the right-hand side of (6.8) (it seems probable that the formula proposed by Anderson (1984) is simply related to this second term (with $a=+\infty$), although we did not try to relate our solution to his).

Remark 3. The formula (6.4) leads to a very simple result in the special case where the source S is $S(r, t-r)=r^{B-k} F(t-r)$ with $F \in C^{\infty}(\mathbb{R})$ being zero in the past, k being (for instance) an integer and B being a complex number. If Re (B) is large enough, the hypotheses of theorem 6.1 are satisfied and formula (6.4) yields:

$$\Box_{\mathrm{R}}^{-1}(\hat{n}^L r^{B-k} F(t-r))=\frac{1}{D(B-k)} \int_{-\infty}^{t-r} \mathrm{d}s\, F(s)\, \hat{\partial}_L\left\{\frac{(t-r-s)^{B-k+l+2}-(t+r-s)^{B-k+l+2}}{r}\right\}, \tag{6.9a}$$

with the denominator

$$D(B-k)=2^{B-k+3}(B-k+2)(B-k+1) \ldots(B-k+2-l). \tag{6.9b}$$

The validity of the formula (6.9) can then be extended to any complex value of B (except maybe for some poles when $B=k-2, k-1, \ldots, k-2+l$ or when $B=k-l-3, k-l-4, \ldots$) by analytic continuation (the analytic continuation of the potentially divergent integral $\int^{t-r} \mathrm{d}s\, F(s)\, \hat{\partial}_L((t-r-s)^{B-k+l+2}/r)$ being, for instance, constructed by integrating by parts).

7. Far-zone structure of the general solution

The asymptotic behaviour of the general MPM metric, within the assumptions of §1, when $r \rightarrow \infty$ at fixed time t is simple because, thanks to the assumption of past stationarity, the metric becomes stationary (and equal to ${}_{\mathrm{S}}g=f+\sum G^n\, {}_{\mathrm{S}}h_n$) as soon as $r \geqslant c(t+T)$. Then it is seen

Downloaded from rsta.royalsocietypublishing.org

from Appendix C, and the works quoted there, that ${}_{\mathrm{S}}h_{\mathrm{part}\,n}$ admits a (truncated) expansion in inverse powers of r of the type:

$$ {}_{\mathrm{S}}h_{\mathrm{part}\,n} = \sum_{k \geqslant n} F_{Qk} \frac{\hat{n}^Q}{r^k}, \tag{7.1}$$

where the F_{Qk} are some constant coefficients. In other words, (7.1) describes the asymptotic behaviour of $\mathscr{g}_{\mathrm{part}}$ at (Minkowskian) *spatial infinity* ($r \to \infty$, t fixed) as well as at (Minkowskian) *past-null infinity* ($r \to \infty$, $t + r/c$ fixed). It remains to study the asymptotic behaviour of $\mathscr{g}_{\mathrm{part}}$ at (Minkowskian) *future null infinity* ($r \to \infty$, at fixed retarded time $u = t - r/c$). It is therefore useful to introduce the following definition of a class of functions of *two* variables which will play the role of 'remainders' in the asymptotic expansions when $r \to \infty$, $u = t - r/c$ fixed:

Definition 7.1. A complex valued function of $\mathbb{R}^2$: $f(r, u)$ is said to be $O^\infty(1/r^N)$ (or to belong to the $O^\infty(1/r^N)$ class) for some positive integer N if the following properties hold:

(*a*) $\exists T$ such that $f(r, u) = 0$ when $u \leqslant -T$,

(*b*) $f(r, u) \in C^\infty(]d, +\infty[\times \mathbb{R})$ for some $d \geqslant 0$,

(*c*) $\partial^{m+k} f / \partial r^m \partial u^k$ is, *uniformly* in u, $O(1/r^{N+m})$ when $r \to \infty$, i.e. $\forall u_0, u_1 \in \mathbb{R}$, $\forall (m, k) \in \mathbb{N}^2$, $\exists M > 0$, $\exists A \geqslant 0$ such that

$$(u_0 \leqslant u \leqslant u_1 \text{ and } r > A) \Rightarrow \left(\left| \frac{\partial^{m+k} f(r, u)}{\partial r^m \partial u^k} \right| < \frac{M}{r^{N+m}} \right).$$

The $O^\infty(1/r^N)$ functions will replace the $O^N(r^N)$ functions when studying the asymptotic behaviour $r \to \infty$, u fixed, instead of the asymptotic behaviour $r \to 0$, t or u fixed. Note that the $O^N(r^N)$ were functions of $\mathbb{R}^4$ instead of $\mathbb{R}^2$. Note also that we have included explicitly in definition 7.1 the condition that the $O(1/r^{N+m})$ bound be *uniform* in u, while in definition 3.1 the uniformity of the $O(r^N)$ bound was a consequence of the other conditions. Finally, the definition 7.1 would be meaningful for $N \in \mathbb{R}^+$.

The following basic stability properties of the classes $O^\infty(1/r^N)$ are easily deduced from the definition 7.1 (we use the same simplified notation as with the $O^N(r^N)$ (see lemma 3.2): $f = O^\infty(1/r^N)$. When needed we mention explicitly the dependence on the two variables: $f(r, u) = O^\infty(1/r^N)(r, u)$).

LEMMA 7.1. *We have*

(i) $O^\infty(1/r^N) + O^\infty(1/r^{N'}) = O^\infty(1/r^{\inf(N, N')})$,

(ii) $O^\infty(1/r^N) \cdot O^\infty(1/r^{N'}) = O^\infty(1/r^{N+N'})$,

(iii) $O^\infty(1/r^N)(r+a, u) = O^\infty(1/r^N)(r, u)$ (*for any real* a),

(iv) $F(u)\, r^a (\lg r)^p\, O^\infty(1/r^N) = O^\infty(1/r^{N-a-\epsilon})$ (*for any* $F(u) \in C^\infty(\mathbb{R})$, $a \leqslant N$, $p \geqslant 0$; *with some* $\epsilon > 0$ *and* $a + \epsilon \in \mathbb{N}$ *if we restrict the definition* 7.1 *to* $N \in \mathbb{N}$),

(v) $\partial^{m+k}/\partial r^m \partial u^k\, O^\infty(1/r^N) = O^\infty(1/r^{N+m})$ (*for any positive integers* m, k),

(vi) $\int_r^{+\infty} \mathrm{d}x\, O^\infty(1/r^N)(x, u) = O^\infty(1/r^{N-1})(r, u)$ (*if* $N > 1$),

(vii) $\int_{-\infty}^{u} \mathrm{d}s\, O^\infty(1/r^N)(r, s) = O^\infty(1/r^N)(r, u)$.

Another important property of the $O^\infty(1/r^N)$ class is its behaviour under the action of the 'regularized' retarded integral operator FP $\Box_{\mathrm{R}}^{-1}$.

Downloaded from rsta.royalsocietypublishing.org

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

LEMMA 7.2. *If* $f(r, u) \in O^{\infty}(1/r^{N+1})$ *with* $N > 1$ (*hence* $N \geqslant 2$ *if we restrict* $N \in \mathbb{N}$) *and is such that* $\hat{n}^L f(r, t-r) \in L^n$ *for some* n, *then there exist a* $C^{\infty}(\mathbb{R})$ *function* $G(u)$, *zero when* $u \leqslant -T$, *and a function* $g(r, u) \in O^{\infty}(1/r^N)$ *such that:*

$$\underset{B=0}{\mathrm{FP}}\ \Box_{\mathrm{R}}^{-1}(r^B \hat{n}^L f(r, t-r)) = \hat{\partial}_L\left(\frac{G(t-r)}{r}\right) + \hat{n}^L g(r, t-r). \tag{7.2a}$$

Symbolically, we can write lemma 7.2 as:

$$N > 1 \Rightarrow \mathrm{FP}\ \Box_{\mathrm{R}}^{-1}(\hat{n}^L O^{\infty}(1/r^{N+1})) = \hat{\partial}_L(G(t-r)/r) + \hat{n}^L O^{\infty}(1/r^N). \tag{7.2b}$$

Proof. If Re(B) is large enough the function $S(r, u) := r^B f(r, u)$ will fulfil the conditions of theorem 6.1. Hence, from (6.8), we find for ${}^B u(\boldsymbol{x}, t) := \Box_{\mathrm{R}}^{-1}(r^B \hat{n}^L f(r, t-r))$:

$${}^B u = \hat{\partial}_L \left\{\frac{{}^B G_a(t-r)}{r}\right\} - \int_{-\infty}^{t-r} \mathrm{d}s\, \hat{\partial}_L \left\{\frac{{}^B R_a[\frac{1}{2}(t+r-s), s]}{r}\right\}, \tag{7.3}$$

with

$${}^B R_a(r, s) := r^l \int_a^r \mathrm{d}x \frac{(r-x)^l}{l!} (2/x)^{l-1} x^B f(x, s), \tag{7.4}$$

and

$${}^B G_a(u) := \int_{-\infty}^{u} \mathrm{d}s\, {}^B R_a[\tfrac{1}{2}(u-s), s]. \tag{7.5}$$

The validity of (7.3) can be extended by analytic continuation (because $\hat{n}^L f \in L^n$) to all $B \in \mathbb{C}'$ $(:= \mathbb{C} - \mathbb{Z})$. If Re$(B) < N-1$ we can choose $a = +\infty$. Then let us define

$$G(u) := \underset{B=0}{\mathrm{FP}}\ {}^B G_{+\infty}(u) = 2 \underset{B=0}{\mathrm{FP}} \int_0^{+\infty} \mathrm{d}x\, {}^B R_{+\infty}(x, u-2x). \tag{7.6}$$

Now the hypothesis $\hat{n}^L f(r, t-r) \in L^n$, after a spherical harmonics projection, implies for $f(r, t)$ an asymptotic expansion when $r \to 0$ of the type $\sum F(t)\, r^a (\lg r)^p$ plus a 'good' remainder ($O(r^K)$ and $C^{K-1}(\mathbb{R}^2_+)$ where $K \in \mathbb{N}$ can be chosen arbitrarily large). It is then easy to check, by standard methods, that $G(u)$ ((7.6)) is $C^{\infty}(\mathbb{R})$ and zero when $u \leqslant -T$. On the other hand the last term (with $a = +\infty$) of (7.3) is analytic near $B = 0$ because we have chosen $N > 1$. Expanding the derivative $\hat{\partial}_L$ by means of formula (A 35*a*) we can write:

$$-\int_{-\infty}^{t-r} \mathrm{d}s\, \hat{\partial}_L \{r^{-1}\, {}^0 R_{+\infty}[\tfrac{1}{2}(t+r-s), s]\} = \hat{n}^L g(r, t-r),$$

where $g(r, u)$ is a sum over i and j (with $0 \leqslant i \leqslant l$, $0 \leqslant j \leqslant l$) of terms of the type

$$r^{-(j+1)} \int_{-\infty}^{u} \mathrm{d}s\, [\tfrac{1}{2}(u-s) + r]^{i+j} \int_{r+\frac{1}{2}(u-s)}^{+\infty} \mathrm{d}x\, x^{-i+1} f(x, s).$$

Using the hypothesis $f \in O^{\infty}(1/r^{N+1})$ with $N > 1$ and lemma 7.1, one finds easily that $g \in O^{\infty}(1/r^N)$. ∎

Having thus defined a 'good' class of 'remainder' terms for an asymptotic expansion at (Minkowskian) future null infinity ($r \to \infty$, $u = t-r$ fixed) we can now introduce a class of functions which will play when $r \to \infty$ (u fixed) the role played by the L^n class when $r \to 0$.

Downloaded from rsta.royalsocietypublishing.org

Definition 7.2. A complex valued function $f(\boldsymbol{x}, t)$ defined in a domain D of $\mathbb{R}^4 (r > r_0$ for some $r_0 \geqslant 0)$ is said to belong to the class of functions $\mathscr{L}^n$ $(n \in \mathbb{N})$ if the following properties hold. For *any* positive integer N there exists a *finite* sum

$$S_N(\boldsymbol{x}, t) = \sum_{p \leqslant n} F_{Qkp}(t-r)\, \hat{n}^Q\, r^{-k}\, (\lg r)^p, \tag{7.7 a}$$

where $k \in \mathbb{N}$, $k \geqslant 1$, $p \in \mathbb{N}$ and $p \leqslant n$, and where the coefficients $F_{Qkp}(u)$ are both $C^\infty(\mathbb{R})$ and zero in the past ($u \leqslant -T$ for some fixed T), such that the difference $f(\boldsymbol{x}, t) - S_N(\boldsymbol{x}, t)$ is a finite sum of terms of the type $\hat{n}^L g(r, t-r)$ where each $g(r, u)$ is $O^\infty\ (1/r^N)$:

$$\forall N, f(\boldsymbol{x}, t) = S_N(\boldsymbol{x}, t) + \sum \hat{n}^L\, O^\infty(1/r^N)\, (r, t-r). \tag{7.7 b}$$

In short we can say that $\mathscr{L}^n$ is the class of functions that admit when $r \to \infty$, $u = t-r$ fixed, an asymptotic expansion to all order N along the scale functions $r^{-k}\ (\lg r)^p$, with $k \geqslant 1$ and $0 \leqslant p \leqslant n$, with coefficients admitting a finite multipolar expansion (the coefficients of which are smooth past-zero functions of u) and with a 'good' $\sum \hat{n}^L\ O^\infty\ (1/r^N)\ (r, u)$ remainder.

Now, the class $\mathscr{L}^0$ is essentially sufficient to describe the far-zone behaviour of the linearized MPM metric $h_{\text{part}\,1}$. Indeed, after (4.7), (2.32), (2.33) and (A 35*a*), $h_{\text{part}\,1}$ can be written as

$$h_{\text{part}\,1} = \sum F_{Qk}(t-r)\, \hat{n}^Q\, r^{-k}, \tag{7.8}$$

which means that the 'dynamic' part of $h_{\text{part}\,1}$ belongs to $\mathscr{L}^0$. As the general MPM metric can be described by the particular MPM metric $g_{\text{part}} = f + \sum G^n\, h_{\text{part}\,n}$ of §4 and that $h_{\text{part}\,n}$ is obtained from the preceding $h_{\text{part}\,m}$s by an algorithm based essentially on algebraic and differential operations and the application of the integral operator $\text{FP}\,\square_{\text{R}}^{-1}$ we need to study the behaviour of the $\mathscr{L}^n$ class under such operations. By using lemma 7.1 it is easy to check the following stability properties of the $\mathscr{L}^n$ classes under algebraic and differential operations.

LEMMA 7.3. *If* $f(\boldsymbol{x}, t) \in \mathscr{L}^n$ *and* $g(\boldsymbol{x}, t) \in \mathscr{L}^m$, *then*

(i) $f(\boldsymbol{x}, t) + g(\boldsymbol{x}, t) \in \mathscr{L}^{\sup(n, m)}$,

(ii) $f(\boldsymbol{x}, t)\, .\, g(\boldsymbol{x}, t) \in \mathscr{L}^{n+m}$,

(iii) $\forall q \in \mathbb{N}, \quad \forall p \in \mathbb{N}, \quad \partial_t^q\, \partial_{i_1 \ldots i_p} f(\boldsymbol{x}, t) \in \mathscr{L}^n$.

Let us now prove the more difficult result.

THEOREM 7.1. *If* $f \in \mathscr{L}^n$ *and* $f \in L^m$ *for some* n *and* m, *and if all the powers of* $1/r$ *appearing in* S_N ((7.7 *a*)) *are of order* $k \geqslant 2$, *then* $\text{FP}\,\square_{\text{R}}^{-1} f \in \mathscr{L}^{n+1}$ (*and* $\in L^{m+1}$ *from theorem* 3.1).

Proof. Let us write for any chosen N, $f = S_{N+1} + R_{N+1}$ with

$$S_{N+1} = \sum_{\substack{p \leqslant n \\ 2 \leqslant k \leqslant N+1}} F_{Qkp}(t-r)\, \hat{n}^Q\, r^{-k}\, (\lg r)^p,$$

$$R_{N+1} = \sum \hat{n}^L\, O^\infty(1/r^{N+1})\, (r, t-r).$$

It is easily proven that $S_{N+1} \in L^n$, thence $R_{N+1} \in L^{\sup(n, m)}$. Applying now lemma 7.2 (or rather a parallel lemma where the condition $\hat{n}^L f \in L^n$ is replaced by $(\sum \hat{n}^L f) \in L^n$), we get

$$\text{FP}\,\square_{\text{R}}^{-1}\, R_{N+1} = \sum \hat{\partial}_L (G(t-r)/r) + \sum \hat{n}^L\, O^\infty(1/r^N).$$

Now as $\text{FP}\,\square_{\text{R}}^{-1} f = FP\,\square_{\text{R}}^{-1}\, S_{N+1} + \text{FP}\,\square_{\text{R}}^{-1}\, R_{N+1}$, it is clear that if we prove that, for all $p \leqslant n$, $\text{FP}\,\square_{\text{R}}^{-1}\, (F(t-r)\, \hat{n}^Q\, r^{-k}\, (\lg r)^p) \in \mathscr{L}^{n+1}$ the theorem will be proven. This will be a consequence of the following lemma.

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

Lemma 7.4. *We have:*

$$\Box_R^{-1}(F(t-r)\, r^{B-2}\, \hat{n}^Q\, (\lg r)^p)|_{B=0} \in \mathscr{L}^{p+1},$$

and if $k \geqslant 3$*:*

$$\underset{B=0}{\mathrm{FP}}\ \Box_R^{-1}(F(t-r)\, r^{B-k}\, \hat{n}^Q\, (\lg r)^p) \in \mathscr{L}^p.$$

(The 'FP' prescription has been dropped in the first relation because the retarded integral converges near $B = 0$.)

The main tool in the proof of lemma 7.4 is the identity

$$\Box_R^{-1}(\hat{n}^Q\, r^{B-k}\, F(t-r)) = -\frac{\hat{n}^Q}{2(B-k+2)}\, r^{B-k+1}\, {}^{(-1)}F(t-r)$$
$$+\frac{(B-k+1-q)\,(B-k+2+q)}{2(B-k+2)}\, \Box_R^{-1}(\hat{n}^Q\, r^{B-k-1}\, {}^{(-1)}F(t-r)), \quad (7.9)$$

where $F \in C^\infty(\mathbb{R})$ is past-zero, and ${}^{(-1)}F$ is its past-zero antiderivative. The identity (7.9) is proven either by integrating by parts the s-integration in (6.9a) or by using the same reasoning as for proving the identity (3.11). Then we can iterate (7.9) (which increases the power of $1/r$ in the last term) and differentiate it p times with respect to B (which adds a factor $(\lg r)^p$). This leads to a formula of the type

$$\Box_R^{-1}(\hat{n}^Q\, r^{B-k} (\lg r)^p\, F(t-r)) = \frac{\partial^p}{\partial B^p}\left\{\sum_{i=1}^{N+2-k} C_i(B)\, \hat{n}^Q\, r^{B-k+2-i}\, {}^{(-i)}F(t-r)\right.$$
$$\left. + D_N(B)\, \Box_R^{-1}(\hat{n}^Q\, r^{B-N-2}\, {}^{(-N-2+k)}F(t-r))\right\}. \quad (7.10)$$

If $k \geqslant 3$ all the coefficients C_i and D_N are analytic at $B = 0$, and if $k = 2$ only the first $q+1$ coefficients C_i have a (simple) pole at $B = 0$. Taking the finite part at $B = 0$ of (7.10) and applying lemma 7.2 to the 'remainder terms' which have precisely the structure FP $\Box_R^{-1}$ $(\hat{n}^Q\, O^\infty\, (1/r^{N+1}))$ (with $O^\infty(1/r^{N+1})$ being a sum of terms of the type $r^{-(N+2)}\, (\lg r)^j\, G(t-r)$), finally proves lemma 7.4 and thereby theorem 7.1. ■

It is to be noted that only the powers $1/r^2$ in the 'source' increase (by one) the number of logarithms in the 'solution' = FP $\Box_R^{-1}$ 'source'. A closer examination of the new logarithms in the solution leads to the relation (with a slightly generalized angular part and $B = 0$ taken from the start because the retarded integral converges anyway)

$$\Box_R^{-1}\, (k^{\alpha_1}\, k^{\alpha_2} \ldots k^{\alpha_l}\, r^{-2}\, (\lg r)^p\, F(t-r)) + (-)^l \frac{(\lg r)^{p+1}}{2(p+1)}\, \partial^{\alpha_1}\, \partial^{\alpha_2} \ldots \partial^{\alpha_l}\left(\frac{1}{r}\, {}^{(-l-1)}F(t-r)\right) \in \mathscr{L}^p, \quad (7.11)$$

where $k^\alpha := (1, n^i)$, $\partial^\alpha := (-\partial_0, \partial_i)$, ${}^{(-1)}F(u) = \int_{-\infty}^{u} \mathrm{d}x\, F(x)$ etc.... In particular, by expanding the derivatives ∂^α and by re-summing the multipolar expansion we find at the 'leading logarithm approximation' when $r \rightarrow +\infty$, $t-r$ fixed,

$$\Box_R^{-1}\, (r^{-2}\, (\lg r)^p\, F(t-r, n^i)) = -\frac{(\lg r)^{p+1}}{2(p+1)\, r} \int_{-\infty}^{t-r} \mathrm{d}s\, F(s, n^i) + O((\lg r)^p/r). \quad (7.12)$$

With lemma 7.3 and theorem 7.1 in hand, it is a simple matter (adapting the proof of theorem 4.1) to prove that the 'dynamic' part of $h_{\mathrm{part}\, n}$ belongs to $\mathscr{L}^{n-1}$, in other words we have the following theorem.

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

THEOREM 7.2. *The far-zone behaviour of* $h_{\text{part}\,n}[M, W](\boldsymbol{x}, t)$ *is described by the following asymptotic expansion (up to an arbitrary order* N*):*

$$h_{\text{part}\,n}(\boldsymbol{x}, t) = \sum_l \hat{n}^L \left\{ \sum_{\substack{0 \leqslant p \leqslant n-1 \\ 1 \leqslant k \leqslant N}} \frac{(\lg r)^p}{r^k} F_{Lkp}(t-r) + R_N^L(r, t-r) \right\}, \tag{7.13}$$

where the functions $F_{Lkp}(u)$ *are* $C^\infty(\mathbb{R})$ *and constant when* $u \leqslant -T$ *and where the 'remainders'* $R_N^L(r, u)$ *are* $O^\infty(1/r^N)$ *(if* N *is large enough; if* N *is smaller than* $n(l_{\max}[M, W]+3)$*, where* $l_{\max}[M, W]$ *is the maximum order of multipolarity in* M *and* W*,* $R_N^L(r, u)$ *will be a 'constant in the past'* $O^\infty(1/r^N)$ *function).*

The appearance of $\lg r/r$ terms in the far-zone behaviour of a radiative metric in harmonic coordinates has been known since the work of Fock (1959) (see also Isaacson & Winicour 1968; Madore 1970). Our result (theorem 7.2) is, in a sense, more general and more precise than Fock's result because it deals with all orders in $1/r$ (and in G) and because it shows how the powers of the logarithms increase with n. However, in another sense, (7.13) is less precise than Fock's result because we do not control which logarithms come from a formal expansion of the multipoles expressed in terms of a 'better' retarded time, $u^* = u - 2GM \lg r$ (with $u = t-r$), according to

$$M_L(u^*) = M_L(u) - 2GM \lg r\, {}^{(1)}M_L(u) + 2G^2M^2 (\lg r)^2\, {}^{(2)}M_L(u) \ldots. \tag{7.14}$$

This problem, as well as the link with the Bondi–Sachs–Penrose approach to the far-zone behaviour, will be considered in more details in sequel papers. For the time being let us only emphasize that, in the present MPM approach, the main interest of the expansion (7.13) lies in its proof where it is tied to the definition of $\mathscr{G}_{\text{part}}$, which means that in this approach, one can *link* the *far-zone* behaviour of a radiative metric to the 'multipole moments', $M_L(u)$, $S_L(u)$, and thereby also to its *near-zone* behaviour (§5).

APPENDIX A. SYMMETRIC TRACE-FREE TENSORS AND MULTIPOLE EXPANSIONS

A 1. *Notation*

We treat the harmonic coordinates (x^0, x^i) $(i = 1, 2, 3)$ as if they were Minkowskian coordinates in flat space. In particular the spatial coordinates x^i are treated as Cartesian coordinates and are raised and lowered by means of the Euclidean metric δ_{ij} = Kronecker delta, so that: $A^i = A_i$, $A_{aa} = A_a{}^a \equiv \sum_a A_a{}^a$. We denote by ϵ_{ijk} the fully antisymmetric Levi–Civita symbol ($\epsilon_{123} = +1$). In order to deal conveniently with sequences of many spatial indices we use an abbreviated notation for 'multi-indices', where an upper-case latin letter denotes a multi-index while the corresponding lower-case letter denotes its number of indices: $L := i_1 i_2 \ldots i_l$; $P := i_1 i_2 \ldots i_p$; $T_Q := T_{i_1 i_2 \ldots i_q}$. When several multi-indices appear simultaneously it is understood that different carrier-letters are used, for instance: $T_{PQ} = T_{i_1 \ldots i_p j_1 \ldots j_q}$. When needed we use also $P-1 := i_1 i_2 \ldots i_{p-1}$ so that the tensor $T_{aP-1} := T_{a i_1 \ldots i_{p-1}}$ has p indices. We denote also $r = ((x_1)^2 + (x_2)^2 + (x_3)^2)^{\frac{1}{2}}$; $n^i = x^i/r$; $\partial_i = \partial/\partial x^i$; $n^L := n^{i_1} \ldots n^{i_l}$; $\partial_L := \partial_{i_1} \ldots \partial_{i_l}$. For any positive integer l we shall denote $l! := l(l-1)\ldots 2.1$; $l!! := l(l-2)\ldots(2$ or $1)$. A multi-summation is always understood for repeated multi-indices: $S_P T^P = S_P T_P = \sum_{i_1, \ldots, i_p} S_{i_1 \ldots i_p} T_{i_1 \ldots i_p}$.

Downloaded from rsta.royalsocietypublishing.org

Given a Cartesian tensor T_P, we denote its symmetric part by parentheses

$$T_{(P)} \equiv T_{(i_1 \dots i_p)} := \frac{1}{p!} \sum_\sigma T_{i_{\sigma(1)} \dots i_{\sigma(p)}} \tag{A 1}$$

(σ running over all permutations of $(12\dots p)$). The symmetric-and-trace-free (STF) part of T_P is denoted indifferently by $\hat{T}_P \equiv T_{\langle P \rangle} \equiv T_{\langle i_1 \dots i_p \rangle}$. The explicit expression of the STF part is (Pirani 1964; Thorne 1980)

$$\hat{T}_P = T_{\langle P \rangle} = \sum_{k=0}^{[\frac{1}{2}p]} a_k^p \, \delta_{(i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \dots i_p) a_1 a_1 \dots a_k a_k}, \tag{A 2a}$$

where

$$S_P = T_{(P)}, \tag{A 2b}$$

$$a_k^p = \frac{p!}{(2p-1)!!} \frac{(-)^k (2p-2k-1)!!}{(p-2k)!\,(2k)!!}, \tag{A 2c}$$

$[\frac{1}{2}p]$ denoting the integer part of $\frac{1}{2}p$. For instance, $\hat{T}_{ij} = T_{(ij)} - \frac{1}{3}\delta_{ij} T_{aa}$; $\hat{T}_{ijk} = T_{(ijk)} - \frac{1}{5}[\delta_{ij} T_{(kaa)} + \delta_{jk} T_{(iaa)} + \delta_{ki} T_{(jaa)}]$.

A 2. *Algebraic reduction of Cartesian tensors*

It is well known (see, for example, Gel'fand *et al.* 1963) that the set of all symmetric trace-free Cartesian tensors of rank l (STF-l tensors) generates an *irreducible* representation of 'weight' l (and dimension $2l+1$) of the group of proper rotations (SO(3)). Any tensor of rank p (member of a reducible tensorial representation of SO(3)) can be decomposed in a sum of algebraically independent pieces each of which belongs to an irreducible representation and, therefore, can be expressed in terms of some STF tensor. More precisely, any tensor T_P can be decomposed in a finite sum of terms of the type $\gamma_P^L \hat{R}_L$ where γ_P^L is a tensor *invariant under* SO(3) (a product of some Kronecker and Levi–Civita symbols) and $\hat{R}_L$ a STF-l tensor ($l \leqslant p$) obtained by contracting T_P by some other invariant tensor $\gamma_L'^P$ (Coope *et al.* 1965, 1970). The highest-weight piece of this decomposition is always $\hat{T}_P$. These assertions are easily proven by induction if one uses the following (straightforwardly checked) formula (which generalizes the well-known $U_i V_j = U_{\langle i} V_{j \rangle} + \frac{1}{2}\epsilon_{aij}(\boldsymbol{U} \times \boldsymbol{V})_a + \frac{1}{3}\delta_{ij}(\boldsymbol{U}.\boldsymbol{V})$):

$$U_i \hat{T}_L = \hat{R}_{iL}^{(+)} + \frac{l}{l+1} \epsilon_{ai\langle i_l} \hat{R}^{(0)}_{i_1 \dots i_{l-1} \rangle a} + \frac{2l-1}{2l+1} \delta_{i \langle i_l} \hat{R}^{(-)}_{i_1 \dots i_{l-1} \rangle}, \tag{A 3}$$

where

$$\hat{R}^{(+)}_{i_1 \dots i_{l+1}} = U_{\langle i_{l+1}} \hat{T}_{i_1 \dots i_l \rangle}, \tag{A 4a}$$

$$\hat{R}^{(0)}_{i_1 \dots i_l} = U_a \hat{T}_{b \langle i_1 \dots i_{l-1}} \epsilon_{i_l \rangle ab}, \tag{A 4b}$$

$$\hat{R}^{(-)}_{i_1 \dots i_{l-1}} = U_a \hat{T}_{a i_1 \dots i_{l-1}}. \tag{A 4c}$$

The well-known law of multiplication of representations $D_s \otimes D_l = D_{|l-s|} \oplus \dots \oplus D_{l+s}$ (which corresponds, in quantum language, to the law of addition of angular momenta: $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$) corresponds, in STF language, to an algebraic reduction of the tensorial product of two STF tensors:

$$\hat{U}_S \hat{V}_L = \sum_{|l-s| \leqslant j \leqslant l+s} \gamma_{SL}^J \hat{R}_J, \tag{A 5}$$

where each weight j appears only once, where γ_{SL}^J is an invariant '$3-j$ tensor', which is separately STF in S, L and J, and where each $\hat{R}_J$ is bilinearly built out of $\hat{U}_S$ and $\hat{V}_L$ (see Coope

Downloaded from rsta.royalsocietypublishing.org

1970). The 'Clebsch–Gordan reduction' (A 5) is easily calculable, for low values of s, from (A 3) and involves only simple numerical coefficients. This simplicity is to be contrasted with the complicated numerical factors which plague the usual angular-momentum theory employed in quantum mechanics (with its apparatus of $3-j$, $6-j$ symbols, etc.). Moreover, Cartesian tensors are more intuitively related to directional properties in three-dimensional space than the canonical basis of irreducible tensors (eigenvectors of J_3) of the usual angular-momentum theory. For these reasons we have followed the numerous authors (notably, in the context of general relativity, K. S. Thorne) who advocate the use of STF tensors.

A 3. *Canonical basis of the vector space of* STF *tensors*

If we denote by $\boldsymbol{e}_i$ ($i = 1, 2, 3$) the Cartesian basis vectors ($e_i^j = \delta_i^j$), it can be easily verified that a basis of the $(2l+1)$-dimensional vector space of STF-l tensors is made out of the STF parts of the l-fold tensorial products $(\boldsymbol{e}_1 + i\boldsymbol{e}_2) \otimes \ldots \otimes (\boldsymbol{e}_1 + i\boldsymbol{e}_2) \otimes \boldsymbol{e}_3 \otimes \ldots \otimes \boldsymbol{e}_3$ (with $i^2 = -1$) and their complex conjugates. More precisely such a basis is $\{\hat{Y}_L^{lm};\ -l \leqslant m \leqslant l\}$ where, when $m \geqslant 0$,

$$\hat{Y}_L^{lm} = A^{lm}\, E_{\langle L\rangle}^{lm}, \qquad \text{(A 6}a\text{)}$$

with

$$E_L^{lm} = (\delta_{i_1}^1 + i\delta_{i_1}^2)\ldots(\delta_{i_m}^1 + i\delta_{i_m}^2)\,\delta_{i_{m+1}}^3 \ldots \delta_{i_l}^3, \qquad \text{(A 6}b\text{)}$$

and

$$A^{lm} = (-)^m\,(2l-1)!!\,[(2l+1)/(4\pi\,(l-m)!\,(l+m)!)]^{\frac{1}{2}}, \qquad \text{(A 6}c\text{)}$$

and when $m < 0$ (the asterisk denoting the complex conjugate)

$$\hat{Y}_L^{lm} = (-)^m\,(\hat{Y}_L^{l|m|})^*. \qquad \text{(A 6}d\text{)}$$

The normalization is such that

$$\sum_{i_1,\ldots,i_l} \hat{Y}_{i_1\ldots i_l}^{lm}\,(\hat{Y}_{i_1\ldots i_l}^{lm'})^* = \delta_{mm'}\,\frac{(2l+1)!!}{4\pi l!}. \qquad \text{(A 7)}$$

The expanded form of $\hat{Y}_L^{lm}$ is (Pirani 1964; Thorne 1980) (for $m \geqslant 0$):

$$\hat{Y}_L^{lm} = (-)^m \left(\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right)^{\frac{1}{2}} \sum_{k=0}^{[\frac{1}{2}(l-m)]} \frac{(-)^k\,(2l-2k-1)!!}{(l-m-2k)!\,(2k)!!}$$
$$\times\,\delta_{(i_1 i_2}\ldots\delta_{i_{2k-1} i_{2k}}(\delta_{i_{2k+1}}^1 + i\delta_{i_{2k+1}}^2)\ldots(\delta_{i_{2k+m}}^1 + i\delta_{i_{2k+m}}^2)\,\delta_{i_{2k+m+1}}^3 \ldots \delta_{i_l)}^3. \qquad \text{(A 6}e\text{)}$$

This basis is linked in a simple manner to the usual scalar spherical harmonics on the unit sphere (normalized so that $\int \mathrm{d}\Omega\, Y^{lm}(Y^{l'm'})^* = \delta_{ll'}\,\delta_{mm'}$ with $\mathrm{d}\Omega = \sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\Phi$):

$$Y^{lm}(\Theta, \Phi) = \hat{Y}_L^{lm}\, n^L = \hat{Y}_L^{lm}\, \hat{n}^L. \qquad \text{(A 8)}$$

A 4. *Multipole expansions and* STF *tensors*

The spherical harmonics expansion of a scalar function f on the unit-sphere S_2, $f(\Theta, \Phi) = \Sigma_{l,m} f_{lm}\, Y^{lm}(\Theta, \Phi)$, can alternatively be written as (with summation over L)

$$f(\boldsymbol{n}) = \sum_{l=0}^{\infty} \hat{f}_L\, \hat{n}^L, \qquad \text{(A 9}a\text{)}$$

where $\hat{f}_L = \Sigma_m f_{lm}\, \hat{Y}_L^{lm}$. The STF tensor coefficients $\hat{f}_L$ in (A 9a) are unique and can be directly computed as the following integrals over S_2 (Thorne 1980):

$$\hat{f}_L = \frac{(2l+1)!!}{4\pi l!} \int \mathrm{d}\Omega(\boldsymbol{n})\, \hat{n}^L\, .f(\boldsymbol{n}). \qquad \text{(A 9}b\text{)}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Downloaded from rsta.royalsocietypublishing.org

Let us now consider a tensor field of 'integer spin s' on the unit sphere, i.e. a STF-s tensor function of $\boldsymbol{n}$: $\hat{T}_S(\boldsymbol{n})$. First each component of $\hat{T}_S$ can be considered as a scalar and expanded along the $\hat{n}^L$s (the validity of such expansions will be discussed in Appendix B):

$$\hat{T}_S(\boldsymbol{n}) = \sum_{l=0}^{\infty} T_{SL}\,\hat{n}^L, \tag{A 10}$$

where the coefficients T_{SL} are separately STF with respect to S and L. Then, thanks to the 'Clebsch–Gordan' reduction (A 5), one can decompose T_{SL} in irreducible pieces $\gamma^J_{SL}\,\hat{R}^{sl}_J$ so that

$$\hat{T}_S(\boldsymbol{n}) = \sum_{l=0}^{\infty} \sum_{|l-s| \leqslant j \leqslant l+s} \gamma^J_{SL}\,\hat{R}^{sl}_J\,\hat{n}^L, \tag{A 11}$$

where $\hat{R}^{sl}_J$ is STF and where γ^J_{SL} is some invariant tensor (made out of δs and ϵs), which is separately STF in S, L and J. The decomposition (A 11) is nothing but an expansion in tensor spherical harmonics in STF guise. The usual canonical-basis form of this expansion is obtained by decomposing each $\hat{R}^{sl}_J$ on the $(2j+1)$-dimensional canonical basis of the STF-j tensors: $\{\hat{Y}^{jm}_J;\ -j \leqslant m \leqslant j\}$. It is convenient to introduce some normalization coefficients $C^{sl,j}$, namely

$$\hat{R}^{sl}_J = C^{sl,j} \sum_{m=-j}^{j} R^{sl,jm}\,\hat{Y}^{jm}_J, \tag{A 12}$$

so that (A 11) can be rewritten as

$$\hat{T}_S(\boldsymbol{n}) = \sum_{j=0}^{\infty} \sum_{l=|j-s|}^{j+s} \sum_{m=-j}^{+j} R^{sl,jm}\,\hat{Y}^{sl,jm}_S(\boldsymbol{n}), \tag{A 13}$$

where

$$\hat{Y}^{sl,jm}_S(\boldsymbol{n}) := C^{sl,j}\gamma^J_{SL}\,\hat{Y}^{jm}_J\,\hat{n}^L. \tag{A 14}$$

We can choose the normalization coefficients so that

$$\int \mathrm{d}\Omega(\boldsymbol{n}) \sum_S \hat{Y}^{sl,jm}_S(\boldsymbol{n})\,(\hat{Y}^{sl',j'm'}_S(\boldsymbol{n}))^* = \delta_{ll'}\,\delta_{jj'}\,\delta_{mm'}. \tag{A 15}$$

The $\hat{Y}^{sl,jm}_S(\boldsymbol{n})$ are the generalization for any integer spin s of the 'pure orbital harmonics' (simultaneous eigenfunctions of $\boldsymbol{L}^2$, $\boldsymbol{J}^2$ and $\boldsymbol{J}_3$) thoroughly studied by Thorne (1980) when $s \leqslant 2$ (remark that Thorne denotes (sl', lm) the superscripts that we have denoted (sl, jm) in accordance with the customary 'quantum' usage: $\boldsymbol{J} = \boldsymbol{L}+\boldsymbol{S}$).

Let us write the explicit form of the STF-tensor spherical harmonics expansion (A 11) for $s = 0, 1, 2$:

$$T(\boldsymbol{n}) = \sum_{j \geqslant 0} \hat{A}_J\,\hat{n}_J, \tag{A 16}$$

$$T_i(\boldsymbol{n}) = \sum_{j \geqslant 0} \hat{B}_J\,\hat{n}_{iJ} + \sum_{j \geqslant 1} \{\hat{C}_{iJ-1}\,\hat{n}_{J-1} + \epsilon_{iab}\,\hat{D}_{bJ-1}\,\hat{n}_{aJ-1}\}, \tag{A 17}$$

$$T_{\langle ik\rangle}(\boldsymbol{n}) = \sum_{j \geqslant 0} \hat{E}_J\,\hat{n}_{ikJ} + \sum_{j \geqslant 1} \{\hat{F}_{J-1\langle i}\,\hat{n}_{k\rangle J-1} + \hat{G}_{bJ-1}\,\epsilon_{ab(i}\,\hat{n}_{k)\,aJ-1}\} + \sum_{j \geqslant 2} \{\hat{H}_{ikJ-2}\,\hat{n}_{J-2} + \epsilon_{ab(i}\,\hat{I}_{k)\,bJ-2}\,\hat{n}_{aJ-2}\}. \tag{A 18}$$

Downloaded from rsta.royalsocietypublishing.org

The general multipole expansion of a symmetric 2-tensor field is obtained by adding (A 18) and δ_{ij} times (A 16). When multipole-expanding a space-time tensor field $\hat{T}_S(x^i, t) \equiv \hat{T}_S(rn^i, t)$ the tensor coefficients of the STF expansion (A 11) become functions of the O(3) invariants r and t: $\hat{R}^{sl}_J(r, t)$. Finally it should be remarked that when dealing with the full O(3) group (including spatial inversions) it is convenient to attribute an 'intrinsic parity' $\pi = \pm 1$ to each tensor (so that $T'_S = \pi . (-)^s\, T_S$ under $x'^i = -x^i$) δ_{ij} being endowed with a positive intrinsic parity, but ϵ_{ijk} with a negative one. Then, if the left-hand sides of (A 16)–(A 18) have $\pi = +1$, the multipole tensor coefficients $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{E}$, $\hat{F}$, $\hat{H}$, will have $\pi = +1$ but $\hat{D}$, $\hat{G}$ and $\hat{I}$ will have $\pi = -1$ (for instance, the mass multipole moments M_L have $\pi = +1$ but the spin multipole moments S_L have $\pi = -1$).

A 5. *A compendium of useful formulae*

Let us gather here, without proofs, some formulae which are of frequent use when dealing with STF multipole expansions. Some of these formulae are taken from (or are equivalent to results of) Thorne (1980), others come from Blanchet (1984). Other formulae are contained in Thorne (1980) which is a basic reference for STF multipole expansions (the latter work uses a notation for multi-indices different from ours: I_l, instead of L, for $i_1 \ldots i_l$). In the first formulae we introduce the special notation (convenient in practical calculations) $A_{\{i_1 \ldots i_l\}}$ for the (un-normalized) sum $\sum_{\sigma \in S} A_{i_{\sigma(1)} \ldots i_{\sigma(l)}}$ where S is the *smallest* set of permutations of $(1 \ldots l)$, which makes $A_{\{i_1 \ldots i_l\}}$ fully symmetrical in $i_1 \ldots i_l$; for instance $\delta_{\{ab}\, n_{c\}} = \delta_{ab}\, n_c + \delta_{bc}\, n_a + \delta_{ca}\, n_b$.

$$\delta_{(i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}}\, n_{i_{2k+1} \ldots i_l)} = \frac{(2k)!\,(l-2k)!}{l!\,(2k-1)!!}\, \delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}}\, n_{i_{2k+1} \ldots i_l\}}; \tag{A 19}$$

$$\hat{n}_L = \sum_{k=0}^{[l/2]} (-)^k \frac{(2l-2k-1)!!}{(2l-1)!!}\, \delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}}\, n_{i_{2k+1} \ldots i_l\}}; \tag{A 20a}$$

$$\hat{\partial}_L = \sum_{k=0}^{[l/2]} (-)^k \frac{(2l-2k-1)!!}{(2l-1)!!}\, \delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}}\, \partial_{i_{2k+1} \ldots i_l\}}\, \Delta^k; \tag{A 20b}$$

$$n_L = \sum_{k=0}^{[l/2]} \frac{(2l-4k+1)!!}{(2l-2k+1)!!}\, \delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}}\, \hat{n}_{i_{2k+1} \ldots i_l\}}; \tag{A 21a}$$

$$\partial_L = \sum_{k=0}^{[l/2]} \frac{(2l-4k+1)!!}{(2l-2k+1)!!}\, \delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}}\, \hat{\partial}_{i_{2k+1} \ldots i_l\}}\, \Delta^k; \tag{A 21b}$$

$$n_i\, \hat{n}_{a_1 \ldots a_l} = \hat{n}_{i a_1 \ldots a_l} + \frac{l}{2l+1}\, \delta_{i\langle a_1}\, \hat{n}_{a_2 \ldots a_l\rangle}; \tag{A 22a}$$

$$n_{PQ}\, \hat{A}_P\, \hat{B}_Q = \sum_{k=0}^{\inf(p,q)} \frac{p!\,q!}{k!\,(p-k)!\,(q-k)!}\, \frac{(2p+2q-4k+1)!!}{(2p+2q-2k+1)!!}\, \hat{n}_{RS}\, \hat{A}_{KR}\, \hat{B}_{KS}, \tag{A 22b}$$

(where $r = p-k$, $s = q-k$);

$$n_i\, \hat{n}_{iL} = \frac{l+1}{2l+1}\, \hat{n}_L; \tag{A 23}$$

$$r \partial_i\, \hat{n}_L = (l+1)\, n_i\, \hat{n}_L - (2l+1)\, \hat{n}_{iL}; \tag{A 24}$$

$$n_L\, \hat{n}'_L = \hat{n}_L\, n'_L = \hat{n}_L\, \hat{n}'_L = \frac{l!}{(2l-1)!!}\, P_l(\boldsymbol{n}.\boldsymbol{n}') \tag{A 25}$$

Downloaded from rsta.royalsocietypublishing.org

where $P_l(x)$ is the usual Legendre polynomial of order l (for a generalization of this formula to the Gegenbauer polynomials see Lucquiaud 1984*a*, *b*);

$$F(\boldsymbol{n}.\boldsymbol{n}') = \frac{1}{2}\sum_{l=0}^{+\infty}\frac{(2l+1)!!}{l!}\hat{n}_L\,\hat{n}'_L\int_{-1}^{+1}\mathrm{d}z\,F(z)\,P_l(z) \tag{A 26}$$

is the expansion of an arbitrary function of $x = \boldsymbol{n}.\boldsymbol{n}'$ in a series of Legendre polynomials;

$$\int \mathrm{d}\Omega\hat{n}_L = 0 \quad \text{if} \quad l \geqslant 1; \tag{A 27}$$

$$\int \mathrm{d}\Omega n_{i_1}\ldots n_{i_{2p+1}} = 0; \tag{A 28a}$$

$$\int \mathrm{d}\Omega n_{i_1}\ldots n_{i_{2p}} = \frac{4\pi}{(2p+1)!!}\delta_{\{i_1 i_2}\ldots\delta_{i_{2p-1}i_{2p}\}}; \tag{A 28b}$$

$$\hat{A}_P\,\hat{B}_Q\int \mathrm{d}\Omega n_{PQ} = \delta_{pq}\frac{4\pi p!}{(2p+1)!!}\hat{A}_P\,\hat{B}_P; \tag{A 29a}$$

$$\hat{A}_P\,\hat{B}_Q\int \mathrm{d}\Omega n_{iPQ} = \delta_{p,\,q+1}\frac{4\pi p!}{(2p+1)!!}\hat{A}_{iQ}\,\hat{B}_Q + \delta_{q,\,p+1}\frac{4\pi q!}{(2q+1)!!}\hat{A}_P\,\hat{B}_{iP}; \tag{A 29b}$$

$$\hat{\partial}_L f(r) = \hat{n}_L\,r^l(r^{-1}\,\partial/\partial r)^l f(r) = \hat{n}_L(2r)^l\,(\partial/\partial(r^2))^l f(r); \tag{A 30}$$

$$\hat{\partial}_L f(r) = \frac{\hat{n}_L}{(-2)^l}\sum_{k=1}^{l}\frac{(-2)^k\,(2l-k-1)!}{(k-1)!\,(l-k)!}r^{k-l}(\partial/\partial r)^k f(r); \tag{A 31}$$

$$\hat{\partial}_L\,r^\lambda = \lambda(\lambda-2)\ldots(\lambda-2l+2)\,\hat{n}_L\,r^{\lambda-l},\ (\forall\lambda\in\mathbb{C}). \tag{A 32}$$

Two particular cases of (A 32) are

$$\hat{\partial}_L\,r^{2j} = 0 \quad \text{if} \quad j = 0,\,1,\,2,\,\ldots,\,l-1, \tag{A 33}$$

$$\hat{\partial}_L\,r^{-1} = \partial_L\,r^{-1} = (-)^l\,(2l-1)!!\frac{\hat{n}_L}{r^{l+1}}. \tag{A 34}$$

$$\hat{\partial}_L\left(\frac{F(t-\epsilon r)}{r}\right) = (-\epsilon)^l\,\hat{n}_L\sum_{j=0}^{l}\frac{(l+j)!}{(2\epsilon)^j j!(l-j)!}\frac{{}^{(l-j)}F\,(t-\epsilon r)}{r^{j+1}}, \quad (\epsilon^2 = 1); \tag{A 35a}$$

$$\hat{\partial}_L\left(\frac{F(t-r)}{r}\right) = \frac{2}{l!}\hat{n}_L(v-u)^l\frac{\partial^{2l}}{\partial u^l\,\partial v^l}\left(\frac{F(u)}{v-u}\right); \tag{A 35b}$$

$$\hat{\partial}_L\left(\frac{F(t+r)}{r}\right) = \frac{2}{l!}\hat{n}_L(v-u)^l\frac{\partial^{2l}}{\partial u^l\,\partial v^l}\left(\frac{F(v)}{v-u}\right), \tag{A 35c}$$

(where $u = t-r$, $v = t+r$);

$$\hat{\partial}_L\left\{\frac{(t+r)^i-(t-r)^i}{r}\right\} = 0 \quad \text{if} \quad i = 0,\,1,\,\ldots,\,2l; \tag{A 36}$$

$$\Delta(r^\lambda\hat{n}_L) = (\lambda-l)\,(\lambda+l+1)\,r^{\lambda-2}\hat{n}_L. \tag{A 37}$$

Downloaded from rsta.royalsocietypublishing.org

Appendix B. Pointwise convergence of multipole expansions

Most text-books of mathematical physics discuss only the convergence *in the quadratic mean* of multipole expansions. An exception is the work of Courant & Hilbert (1953, volume I, p. 513), which discusses the pointwise convergence of the usual *scalar* spherical harmonics expansion, although by means of quite indirect arguments. In the MPM approach to gravitational radiation theory one needs to perform many nonlinear pointwise operations on tensor spherical harmonics expansions, it is therefore useful to have a good direct control of the *pointwise* convergence of *tensor* spherical harmonics expansions.

It has been pointed out to one of us in a personal communication by B. Simon (1984) that a useful identity to control the pointwise convergence of scalar spherical harmonics expansions is

$$\sum_{m=-l}^{+l} |Y^{lm}(\Theta, \Phi)|^2 = \frac{2l+1}{4\pi}. \tag{B 1}$$

We shall first generalize the identity (B 1) to the tensorial case. We have introduced in Appendix A an orthonormal set of STF-s tensor spherical harmonics: $\{\hat{Y}^{sl,jm}_{i_1\ldots i_s}(\boldsymbol{n}); j \geqslant 0, -j \leqslant m \leqslant j, |j-s| \leqslant l \leqslant j+s\}$. It is easily seen from the definition (A 14) (where γ^J_{SL} is a location-independent invariant tensor) that under a proper (active) rotation R the $(2j+1)$ tensor fields $\{\hat{Y}^{sl,jm}_S(\boldsymbol{n}); -j \leqslant m \leqslant j\}$ transform as

$$R(\hat{Y}^{sl,jm}_S)\,(\boldsymbol{n}) = C^{sl,j}\,\gamma^J_{SL}\,\hat{n}^L\,R(\hat{Y}^{jm}_J) \tag{B 2}$$

(the rotation $R = (R_{ij})$ acts both on the spin indices S and the field point $\boldsymbol{n}$, hence the transform of a tensor field is $R(T_{i_1\ldots i_s})\,(\boldsymbol{n}) = R_{i_1 a_1}\ldots R_{i_s a_s}\,T_{a_1\ldots a_s}(R^{-1}(\boldsymbol{n}))$, where $R^{-1}(\boldsymbol{n})_i = R_{ai}\,n_a$). By definition, the canonical basis $\{\hat{Y}^{jm}_J; -j \leqslant m \leqslant j\}$ of the $(2j+1)$-dimensional set of STF-j (location-independent) tensors generates a *unitary* representation (of weight j) of SO(3); i.e. there is a unitary matrix $D^{m}_{(j)\;m'}(R)$ such that

$$R(\hat{Y}^{jm}_J) = \sum_{m'=-j}^{+j} D^{m}_{(j)\;m'}(R)\,\hat{Y}^{jm'}_J. \tag{B 3}$$

Hence the $(2j+1)$ tensor *fields* $\hat{Y}^{sl,jm}_S(\boldsymbol{n})$ $(-j \leqslant m \leqslant j)$ transform under the same unitary transformation:

$$R(\hat{Y}^{sl,jm}_S)\,(\boldsymbol{n}) = \sum_{m'=-j}^{+j} D^{m}_{(j)\;m'}(R)\,\hat{Y}^{sl,jm'}_S(\boldsymbol{n}). \tag{B 4}$$

Now (B 4) and the unitary character of the matrix $D_{(j)}$ imply that the scalar field

$$f^{sl,j}(\boldsymbol{n}) = \sum_S \sum_{m=-j}^{+j} |\hat{Y}^{sl,jm}_S(\boldsymbol{n})|^2 \tag{B 5}$$

is *invariant* under a rotation $(R(f)\,(\boldsymbol{n}) \equiv f(R^{-1}\boldsymbol{n}) = f(\boldsymbol{n}))$. Therefore $f^{sl,j}(\boldsymbol{n})$ is constant on S_2 and the value of the constant is easily obtained from the orthonormality relations (A 15). This leads to the following generalization of (B 1):

$$\sum_{i_1,\ldots,i_s} \sum_{m=-j}^{+j} |\hat{Y}^{sl,jm}_{i_1\ldots i_s}(\boldsymbol{n})|^2 = \frac{2j+1}{4\pi}. \tag{B 6}$$

Downloaded from rsta.royalsocietypublishing.org

Let us consider a STF-s tensor field on the unit-sphere ($\boldsymbol{n}^2 = 1$), say $T_S(\boldsymbol{n})$, and its associated (formal) expansion in tensor–spherical harmonics:

$$\sum_{j=0}^{\infty} \sum_{l=|j-s|}^{j+s} H_S^{sl,j}(\boldsymbol{n}), \tag{B 7}$$

where

$$H_S^{sl,j}(\boldsymbol{n}) := \sum_{m=-j}^{+j} A^{sl,jm}\, \hat{Y}_S^{sl,jm}(\boldsymbol{n}), \tag{B 8}$$

with

$$A^{sl,jm} := \sum_S \int_{S_2} \mathrm{d}\Omega(\boldsymbol{n})\, (\hat{Y}_S^{sl,jm}(\boldsymbol{n}))^*\, T_S(\boldsymbol{n}). \tag{B 9}$$

At this point we assume only that $T_S(\boldsymbol{n})$ is regular enough (e.g. continuous) for the integrals (B 9) to exist. Then because of Bessel's inequality (valid for any orthonormal system, see Courant & Hilbert 1953, p. 51) we already know that the series $\sum_{j,l,m} |A^{sl,jm}|^2$ converges. This result is, however, too weak to ensure the pointwise convergence of the series (B 7).

LEMMA B 1. *If $T_S(\boldsymbol{n})$ is a twice continuously differentiable* STF-s *Cartesian tensor field on the unit-sphere $S_2(\boldsymbol{n}^2 = 1)$, then there exists a numerical sequence ϵ_j tending to zero when $j \to \infty$, such that each 'harmonic' piece of the tensor spherical harmonic expansion* (B 7) *of T_S admits the following uniform bound on S_2 (for any $j \geqslant 1$ and l with $|j-s| \leqslant l \leqslant j+s$):*

$$\left(\sum_S |H_S^{sl,j}(\boldsymbol{n})|^2\right)^{\frac{1}{2}} \leqslant \epsilon_j / j^{\frac{3}{2}}. \tag{B 10}$$

Proof. The infinitesimal generator of rotations acting on the tensor field $T_S(\boldsymbol{n})$ is a first-order differential operator $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ (where $\boldsymbol{L} = -\mathrm{i}\boldsymbol{r} \times \boldsymbol{\partial}$ acts tangentially to S_2 and $\boldsymbol{S}$ is a matrix acting on the spin indices). Hence $(\boldsymbol{J})^2$ is a second-order differential operator on S_2, so that by hypothesis the (STF) tensor field $\tilde{T}_S(\boldsymbol{n}) := (\boldsymbol{J})^2\, T_S(\boldsymbol{n})$ is continuous on S_2. We can then consider the multipole expansion of $\tilde{T}_S(\boldsymbol{n})$. As $(\boldsymbol{J})^2$ is self-adjoint, the expansion coefficients $\tilde{A}^{sl,jm}$ of $\tilde{T}_S(\boldsymbol{n})$ (equation (B 9)) will be equal to $\sum_S \int \mathrm{d}\Omega((\boldsymbol{J})^2\, \hat{Y}_S^{sl,jm})^*\, T_S$. As $\hat{Y}_S^{sl,jm}(\boldsymbol{n})$ generates an irreducible representation of the rotation group of weight j (equation (B 4)), it is an eigenfunction of $(\boldsymbol{J})^2$ with eigenvalue $j(j+1)$. Therefore

$$\tilde{A}^{sl,jm} = j(j+1)\, A^{sl,jm}. \tag{B 11}$$

Now by Schwarz inequality (with respect to $\sum_m$) followed by a summation over S, we deduce from (B 8) that

$$\sum_S |H_S^{sl,j}(\boldsymbol{n})|^2 \leqslant \left(\sum_{m=-j}^{+j} |A^{sl,jm}|^2\right)\left(\sum_S \sum_{m=-j}^{+j} |\hat{Y}_S^{sl,jm}(\boldsymbol{n})|^2\right). \tag{B 12}$$

Using now (B 6) and (B 11) we get ($j \geqslant 1$),

$$\sum_S |H_S^{sl,j}(\boldsymbol{n})|^2 \leqslant \left(\sum_{m=-j}^{+j} |\tilde{A}^{sl,jm}|^2\right)(2j+1)/(4\pi j^2 (j+1)^2). \tag{B 13}$$

Because of Bessel's inequality (applied to $\tilde{T}_S$) the series $\sum_{j,l,m} |\tilde{A}^{sl,jm}|^2$ converges, thus $\sum_{l=|j-s|}^{l=j+s} \sum_{m=-j}^{m=+j} |\tilde{A}^{sl,jm}|^2$ tends to zero when j tends to infinity. Hence (B 13) implies (B 10). ∎

Note that if $T_S(\boldsymbol{n})$ is $C^{2n}(S_2)$ a similar reasoning (using $(\boldsymbol{J})^{2n}$) leads to a faster uniform decrease of the 'harmonic pieces' ($\leqslant \epsilon_j / j^{\frac{1}{2}(4n-1)}$). Let us now use the bound (B 10) to prove that the tensor spherical harmonics series (B 7) converges pointwise to $T_S(\boldsymbol{n})$.

Downloaded from rsta.royalsocietypublishing.org

THEOREM B 1. *Any twice continuously differentiable* STF-*s Cartesian tensor field on the unit sphere*, $T_S(\boldsymbol{n})$, *can be pointwise expanded in a tensor spherical harmonics series* (*namely* (B 7)) *which is uniformly convergent on the unit sphere.*

Proof. As the right-hand side of (B 10) is a convergent numerical series, we first conclude that the series (B 7) must converge pointwise to some tensor field, say $\overline{T}_S(\boldsymbol{n})$. Moreover, as each 'harmonic piece' $H_S^{sl,j}(\boldsymbol{n})$ is continuous on S_2, and as the convergence to $\overline{T}_S(\boldsymbol{n})$ is uniform on S_2 (because the bound (B 10) is uniform) we deduce from standard theorems on uniform convergence that the limit field $\overline{T}_S(\boldsymbol{n})$ is continuous on S_2. Therefore, if we show (denoting $\|U_S\|^2_{L^2} := \int \mathrm{d}\Omega \,\Sigma_S |U_S(\boldsymbol{n})|^2$) that

$$\| T_S - \overline{T}_S \|^2_{L^2} = 0, \tag{B 14}$$

the continuity of $T_S(\boldsymbol{n})$ and $\overline{T}_S(\boldsymbol{n})$ will imply that $\forall \boldsymbol{n}$, $T_S(\boldsymbol{n}) = \overline{T}_S(\boldsymbol{n})$ and the theorem will be proved. Now, (B 14) will be true if we only prove that

$$\operatorname*{limit}_{k\to\infty} \left\| T_S - \sum_{j=0}^{k} \sum_{l=|j-s|}^{j+s} H_S^{sl,j} \right\|^2_{L^2} = 0 \tag{B 15}$$

(indeed, as we have shown that (B 7) is *uniformly* convergent we can go to the limit inside the integral in (B 15)). By well-known reasonings (see Courant & Hilbert 1953, p. 51), equation (B 15) is equivalent to saying that the set of tensor spherical harmonics is (L^2) '*complete*', i.e. that any continuous STF tensor field on S_2 can be approximated in the quadratic mean, with any prescribed accuracy, by some finite linear combination of the $\hat{Y}_S^{sl,jm}$. Finally this completeness follows from Weierstrass's approximation theorem, which implies that the continuous field $r\,.\,T_S(\boldsymbol{n})$ can be uniformly approximated in the cube $-2 \leqslant x^i \leqslant 2$ by polynomials in x^i ($i = 1, 2, 3$) and therefore that $T_S(\boldsymbol{n})$ can be uniformly approximated on S_2 by polynomials in n^i that, by (A 21*a*) and the reduction (A 10)–(A 11), can be written as some finite linear combination of the $\hat{Y}_S^{sl,jm}$. ∎

APPENDIX C. STATIONARY MPM METRICS

It is known that stationary asymptotically flat space–times are analytic in a neighbourhood of spatial infinity (Beig & Simon 1981; Beig 1981) and thus that there exists a 'good' class of coordinate systems in which the metric coefficients admit expansions when $r \to \infty$ in powers of $1/r$ (without $\lg r$). These expansions are uniquely determined in the conformal space by the Geroch–Hansen multipole moments (Geroch 1970; Hansen 1974) or, in the physical space, by the Thorne (1980) multipole moments or the Simon–Beig (1983) ones (see Gürsel 1983; Simon & Beig 1983 for equivalence between these various moments). We wish here to recover, and to construct explicitly, these expansions within the framework of MPM metrics, using harmonic coordinates in physical space and the Thorne moments. This will prove that harmonic coordinates belong to the 'good' class of coordinates. More details about what follows are contained in Blanchet (1984).

C 1. *Construction*

We recursively assume that some 'particular' $h^{\alpha\beta}_{\mathrm{part}\,m}[{}_{\mathrm{S}}M, {}_{\mathrm{S}}W]$, for $m < n$, are constructed which admit a finite expansion in powers of $1/r$ of the type

$$h^{\alpha\beta}_{\mathrm{part}\,m}[{}_{\mathrm{S}}M, {}_{\mathrm{S}}W] = \sum_{q,\,k} F_{Qk} \frac{\hat{n}^Q}{r^k}, \tag{C 1}$$

Downloaded from rsta.royalsocietypublishing.org

where F_{Qk} is a contracted product of 0 or 1 Levi–Civita tensor, p Kronecker tensors and m stationary STF tensors chosen among the ${}_{\mathrm{S}}M = \{M_L, S_L\}$ or ${}_{\mathrm{S}}W = \{W_L, X_L, Y_L, Z_L\}$ which generate $h_{\mathrm{part}\,1}[{}_{\mathrm{S}}M, {}_{\mathrm{S}}W]$ ((4.7) with (2.32) and (2.33)). Replacing the $h_{\mathrm{part}\,m}$s into $N_n(h_m; m < n)$ leads to a finite sum

$$N^{\alpha\beta}_{\mathrm{part}\,n}[{}_{\mathrm{S}}M, {}_{\mathrm{S}}W] = \sum_{q,k} G_{Qk} \frac{\hat{n}^Q}{r^{k+2}}, \tag{C 2}$$

where G_{Qk} has the same structure as F_{Qk}. If $\Sigma^n_{i=1} l_i$ is the total number of indices on the n tensors $M_{L_1}, \ldots, Z_{L_n}$ composing G_{Qk} and if a, b and d are respectively the numbers of W_Ps, X_Qs and Z_Rs among the $n\,M_{L_1}, \ldots, Z_{L_n}$, then (by a dimensional argument):

$$k = n + \sum_{i=1}^{n} l_i + a + 2b + d. \tag{C 3}$$

In the following we will need the inequality

$$\sum_{i=1}^{n} l_i - q + s \geqslant 0 \tag{C 4}$$

relating $\Sigma^n_{i=1} l_i$ and q with the number s of spatial indices on $N^{\alpha\beta}_{\mathrm{part}\,n}$ ($s = 0, 1, 2$ according to $\alpha\beta = 00, 0i, ij$). This inequality can be proven by an argument of 'addition of angular momenta' (Thorne, personal communication 1984) or equivalently by equating the number of free (i.e. non-contracted) spatial indices on both sides of (C 2).

Using the function $B \rightarrow \Delta^{-1}(\hat{n}^Q r^{B+a})$ (analytic in $\mathbb{C}' = \mathbb{C} - \mathbb{Z}$) of (3.9) we define

$$h^{\alpha\beta}_{\mathrm{part}\,n}[{}_{\mathrm{S}}M, {}_{\mathrm{S}}W] := \sum_{q,k} G_{Qk} \operatorname*{FP}_{B=0} \Delta^{-1}(\hat{n}^Q r^{B-k-2}). \tag{C 5}$$

Then $h^{\alpha\beta}_{\mathrm{part}\,n}$ solves Einstein's stationary equations. Indeed, first $\Delta h^{\alpha\beta}_{\mathrm{part}\,n} = N^{\alpha\beta}_{\mathrm{part}\,n}$ is checked to be true (adapting the proof of (3.15)); second, $\partial_\beta h^{\alpha\beta}_{\mathrm{part}\,n}$ is a sum of terms

$$G_{Qk} \cdot \operatorname*{Residue}_{B=0} \left\{ \frac{r^{B-k-1} \hat{n}^Q}{(B-k-1-q)\,(B-k+q)} \right\}$$

(similarly to (4.10)) which are zero unless $q = k$. But, by (C 3) and (C 4) (with $s = 0$ or 1), we have: $q = k \Rightarrow n \leqslant 1$. Therefore: $\partial_\beta h^{\alpha\beta}_{\mathrm{part}\,n} = 0$, as was to be proven.

However, it is necessary to prove that each function

$$B \rightarrow \Delta^{-1}(\hat{n}^Q r^{B-k-2}) = \frac{\hat{n}^Q r^{B-k}}{(B-k-q)\,(B-k+1+q)}$$

in (C 5) is well defined for $B = 0$ (no pole at $B = 0$); because if it is not, taking the finite part will produce a $\lg r$ and our recursive assumption (C 1) will fail at order n. A pole arises when $q = k - 1$, which implies, thanks to (C 3) and (C 4),

$$n = q - \sum_{i=1}^{n} l_i + 1 - a - 2b - d \leqslant s + 1 - a - 2b - d, \tag{C 6}$$

so that necessarily $n \leqslant 3$ (Thorne, personal communication 1984; correcting section X (ii) of Thorne 1980). In the quadratic case $n = 2$ we easily see, thanks to the structure of $N_{\mathrm{part}\,2}$ ($\sim \partial_P r^{-1} \partial_Q r^{-1}$) that $B \rightarrow \Delta^{-1}(r^B N_{\mathrm{part}\,2})$ has no pole in $B = 0$. In the cubic case $n = 3$, the

Downloaded from rsta.royalsocietypublishing.org

'critical' terms in $N_{\text{part}\,3}$, which generate a pole, are such that $s = 2$, $a = b = d = 0$ (see (C 6)) and $q = l_1+l_2+l_3+2$. We find

$$(N^{ij}_{\text{part}\,3})_{\text{critical}} = \sum_{l_1,\,l_2,\,l_3} \frac{\hat{n}_{ijL_1L_2L_3}}{r^{l_1+l_2+l_3+5}}\{\tilde{A}M_{L_1}M_{L_2}M_{L_3}+\tilde{B}M_{L_1}S_{L_2}S_{L_3}$$
$$+\tilde{C}Y_{L_1}Y_{L_2}Y_{L_3}+\tilde{D}M_{L_1}Y_{L_2}Y_{L_3}+\tilde{E}M_{L_1}M_{L_2}Y_{L_3}+\tilde{F}S_{L_1}S_{L_2}Y_{L_3}\}, \quad \text{(C 7)}$$

(note the even number of current multipoles S_L), where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{F}$ are constant coefficients that we shall prove to be *all* zero; $\tilde{A} = \tilde{B} = \tilde{C} = \tilde{D} = \tilde{E} = \tilde{F} = 0$.

(i) *Proof that* ${}_{MMM}N^{ij}_3 \equiv 0$ (*hence* $\tilde{A} = 0$). Two remarkable facts allow us to prove that ${}_{MMM}N^{ij}_3$ (the part of $N^{ij}_{\text{part}\,3}$ which is composed by three interacting mass multipoles M_L) is identically zero (not only the 'critical' part is zero). First, ${}_{MM}h^{00}_2$ and $\Sigma_i\, {}_{MM}h^{ii}_2 = \delta_{ij}\, {}_{MM}h^{ij}_2$ (the $M_{L_1}\times M_{L_2}$ parts of $h_{\text{part}\,2} = \Delta^{-1}(r^B N_{\text{part}\,2})_{|B=0}$) satisfy (${}_M h^{00}_1$ being the part of $h^{00}_{\text{part}\,1}$ composed with M_L)

$$\Delta_{MM}h^{00}_2 = {}_{MM}N^{00}_2 = -\tfrac{7}{8}\partial_k\, {}_M h^{00}_1\, \partial_k\, {}_M h^{00}_1, \quad \text{(C 8a)}$$

$$\sum_i \Delta\, {}_{MM}h^{ii}_2 = \sum_i {}_{MM}N^{ii}_2 = -\tfrac{1}{8}\partial_k\, {}_M h^{00}_1\, \partial_k\, {}_M h^{00}_1. \quad \text{(C 8b)}$$

In (C 8) and in the following we use for computations of the 'sources' N_2 and N_3 the Appendix A of Bel *et al.* (1981). From $\partial_k h_1\, \partial_k h_1 = \frac{1}{2}\Delta(h^2_1)$ and the structure of h_1 ($\sim \partial_L r^{-1}$) we can prove

$$\Delta^{-1}(r^B\, \partial_k h_1\, \partial_k h_1)_{|B=0} = \tfrac{1}{2}h^2_1, \quad \text{(C 9)}$$

which gives

$${}_{MM}h^{00}_2 = -\tfrac{7}{16}({}_M h^{00}_1)^2, \quad \text{(C 10a)}$$

$$\sum_i {}_{MM}h^{ii}_2 = -\tfrac{1}{16}({}_M h^{00}_1)^2. \quad \text{(C 10b)}$$

Second ${}_{MMM}N^{ij}_3$ involves only terms with ${}_M h^{\alpha\beta}_1$, ${}_{MM}h^{00}_2$ and $\Sigma_i\, {}_{MM}h^{ii}_2$ and does not depend on ${}_{MM}h^{ij}_2$:

$${}_{MMM}N^{ij}_3 = \tfrac{1}{2}\partial_{(i}\, {}_M h^{00}_1\, \partial_{j)}\, ({}_{MM}h^{00}_2+\sum_m {}_{MM}h^{mm}_2) - \tfrac{1}{4}\delta_{ij}\,\partial_k\, {}_M h^{00}_1\, \partial_k({}_{MM}h^{00}_2+\sum_m {}_{MM}h^{mm}_2)$$
$$+\tfrac{1}{2}\, {}_M h^{00}_1\, \partial_i\, {}_M h^{00}_1\, \partial_j\, {}_M h^{00}_1 - \tfrac{1}{4}\delta_{ij}\, {}_M h^{00}_1\, \partial_k\, {}_M h^{00}_1\, \partial_k\, {}_M h^{00}_1. \quad \text{(C 11)}$$

(by using ${}_M h^{0i}_1 = {}_M h^{ij}_1 = 0$). Now, replacing in (C 11) ${}_{MM}h^{00}_2+\Sigma_m\, {}_{MM}h^{mm}_2$ by $-\frac{1}{2}({}_M h^{00}_1)^2$ (equations (C 10)) leads to ${}_{MMM}N^{ij}_3 \equiv 0$ (in particular $\tilde{A} = 0$). Note that, since the mass multipoles M_L are arbitrary, any *static* metric will have the latter property (in harmonic coordinates). For instance, this is true for the Schwarzschild metric for which we have $h^{ij}_n = 0$ for $n \geqslant 3$.

(ii) *Proof that* $\tilde{B} = 0$. In order to reach the critical ${}_{MSS}N^{ij}_3$, we must control the terms in ${}_{SS}h^{00}_2+\Sigma_i\, {}_{SS}h^{ii}_2$ and ${}_{MS}h^{0i}_2$ which have 'maximum multipolarity' (i.e. terms $\sim \hat{n}_L$ with $l = l_1+l_2$ for interactions $M_{L_1}\times S_{L_2}$ or $S_{L_1}\times S_{L_2}$). We replace ${}_M h^{00}_1 = -4\Sigma_{l\geqslant 0}\,(-)^l/l!\,(\partial_L r^{-1})\, M_L$ and ${}_S h^{0i}_1 = 4\,\Sigma_{l\geqslant 1}\,(-)^l\, l/(l+1)!\,\epsilon_{iab}\,(\partial_{aL-1} r^{-1})\, S_{bL-1}$ into

$${}_{SS}N^{00}_2+\sum_i {}_{SS}N^{ii}_2 = 2\partial_k\, {}_S h^{0m}_1\, \partial_m\, {}_S h^{0k}_1, \quad \text{(C 12a)}$$

$${}_{MS}N^{0i}_2 = \partial_k\, {}_M h^{00}_1(\partial_i\, {}_S h^{0k}_1 - \partial_k\, {}_S h^{0i}_1), \quad \text{(C 12b)}$$

Downloaded from rsta.royalsocietypublishing.org

we keep only the terms $\sim \hat{n}_{L_1 L_2}$ and apply the operator $\Delta^{-1}(r^B\ldots)_{|B=0}$. This leads to

$$_{SS}h_2^{00}+\sum_i {}_{SS}h_2^{ii} = \sum_{l_1,\,l_2} \frac{-32 l_1 l_2 (2l_1-1)!!\,(2l_2-1)!!}{(l_1+1)!\,(l_2+1)!} \frac{\hat{n}_{L_1L_2}}{r^{l_1+l_2+2}} S_{L_1} S_{L_2}+\ldots, \qquad \text{(C 13}a\text{)}$$

$$_{MS}h_2^{0i} = \sum_{l_1,\,l_2} \frac{8l_2(2l_1+1)!!\,(2l_2+1)!!}{l_1!\,(l_2+1)!\,(l_1+l_2+1)} \frac{\epsilon_{iab}}{r^{l_1+l_2+2}}$$
$$\times\left(\frac{l_1}{2l_1+1}\hat{n}_{bL_1-1\,L_2} M_{aL_1-1} S_{L_2}+\frac{l_2}{2l_2+1}\hat{n}_{aL_1\,L_2-1} M_{L_1} S_{bL_2-1}\right)+\ldots. \qquad \text{(C 13}b\text{)}$$

(Note that $_{SS}h_2^{00}$ and $\sum_i {}_{SS}h_2^{ii}$, unlike $_{MS}h_2^{0i}$, could be computed exactly by the same method as the one employed for (C 10)). Now the only pieces in $_{MSS}N_3^{ij}$ which will contribute to 'critical' terms are those of the type $\partial_i h_1 \partial_j h_2$ or $h_1 \partial_i h_1 \partial_j h_1$, i.e.

$$_{MSS}N_3^{ij} = \tfrac{1}{2}\partial_{(i\;M}h_1^{00}\,\partial_{j)}\,({}_{SS}h_2^{00}+\sum_i {}_{SS}h_2^{ii}) - 2\,\partial_{(i\,S}h_1^{0k}\partial_{j)\;MS}h_2^{0k}$$
$$-{}_Mh_1^{00}\,\partial_i\,{}_Sh_1^{0k}\,\partial_j\,{}_Sh_1^{0k} - {}_Sh_1^{0k}\,\partial_{(i\;S}h_1^{0k}\,\partial_{j)\;M}h_1^{00}+\ldots. \qquad \text{(C 14)}$$

Plugging (C 13) into (C 14), we readily find $({}_{MSS}N_3^{ij})_{\text{critical}} = 0$, hence $\tilde{B} = 0$.

(iii) *Proof that* $\tilde{C} = \tilde{D} = \tilde{E} = \tilde{F} = 0$. We have just shown that the 'canonical' stationary metric (that is, the 'particular' metric with $W_L = X_L = Y_L = Z_L = 0$) satisfies (C 1) (no lg r). Now we extend this result to the particular metric by showing that $N_{\text{part}\,3}-N_{\text{can}\,3}$ is a sum of 'non-critical' terms. To do that we perform upon the canonical metric the non-harmonic coordinate transformation $x'^\alpha = x^\alpha + Gw^\alpha\,[{}_SW]$ (where $w^\alpha\,[{}_SW]$ is the vector of (2.33)). Then $h_{\text{can}\,1}\,[{}_SM]$ is transformed into $h_{\text{part}\,1}\,[{}_SM,\,{}_SW]$ (by (4.7)), and $h_{\text{can}\,2}\,[{}_SM]$ and $h_{\text{can}\,3}\,[{}_SM]$ are transformed into

$$h_2'\,[{}_SM,\,{}_SW] = h_{\text{can}\,2}\,[{}_SM]+k_2\,[{}_SM,\,{}_SW], \qquad \text{(C 15}a\text{)}$$

$$h_3'\,[{}_SM,\,{}_SW] = h_{\text{can}\,3}\,[{}_SM]+k_3\,[{}_SM,\,{}_SW], \qquad \text{(C 15}b\text{)}$$

where k_2 and k_3 are sums of terms of the following type:

$$k_2\,[{}_SM,\,{}_SW] \sim h_{\text{can}\,1}\,\partial w + w\,\partial h_{\text{can}\,1} + \partial w\,\partial w, \qquad \text{(C 16}a\text{)}$$

$$k_3\,[{}_SM,\,{}_SW] \sim h_{\text{can}\,2}\,\partial w + w\,\partial h_{\text{can}\,2} + h_{\text{can}\,1}\,\partial w\,\partial w + ww\,\partial\partial h_{\text{can}\,1} + \partial w\,\partial ww. \qquad \text{(C 16}b\text{)}$$

We have included in k_2 and k_3 the necessary terms such that (C 15) are functional equalities, that is that both sides of (C 15) are computed at the same values of the coordinates. The divergences of h_2' and h_3' are

$$\partial_\beta h_2'^{\alpha\beta}\,[{}_SM,\,{}_SW] = 0, \qquad \text{(C 17}a\text{)}$$

$$\partial_\beta h_3'^{\alpha\beta}\,[{}_SM,\,{}_SW] = h^{ij}_{\text{can}\,2}\,[{}_SM]\,\partial_{ij}\,w^\alpha\,[{}_SW], \qquad \text{(C 17}b\text{)}$$

where we have used $h^{ij}_{\text{can}\,1}\,[{}_SM] = 0$. Writing Einstein's (harmonic) equations for $h_2'^{\alpha\beta}$ then shows (via (C 15*a*)) that

$$N_2^{\alpha\beta}(h_{\text{part}\,1}\,[{}_SM,\,{}_SW]) = N_2^{\alpha\beta}(h_{\text{can}\,1}\,[{}_SM]) + \Delta k_2^{\alpha\beta}\,[{}_SM,\,{}_SW], \qquad \text{(C 18)}$$

and thus

$$h^{\alpha\beta}_{\text{part}\,2}\,[{}_SM,\,{}_SW] = h^{\alpha\beta}_{\text{can}\,2}\,[{}_SM] + \Delta^{-1}(r^B\,\Delta k_2^{\alpha\beta}\,[{}_SM,\,{}_SW])_{|B=0}. \qquad \text{(C 19)}$$

Downloaded from rsta.royalsocietypublishing.org

Now, thanks to the structure (C 16*a*) of k_2 ($\sim \partial_P r^{-1} \partial_Q r^{-1}$), we find (similarly to (C 9)) $\Delta^{-1}(r^B \Delta k_2^{\alpha\beta})_{|B=0} = k_2^{\alpha\beta}$, and therefore $h'_2[{}_{\rm S}M, {}_{\rm S}W] = h_{\text{part }2}[{}_{\rm S}M, {}_{\rm S}W]$. Consider $h'_3[{}_{\rm S}M, {}_{\rm S}W]$: it satisfies Einstein's non-harmonic equations (see (4.2)–(4.4)) with 'source' $N_{\text{part }3}$ (since $h'_1 = h_{\text{part }1}$ and $h'_2 = h_{\text{part }2}$). Thus we have the looked-for relation (using (C 15*b*) and (C 17*b*)):

$$N^{\alpha\beta}_{\text{part }3} - N^{\alpha\beta}_{\text{can }3} = -\partial^\alpha(h^{ij}_{\text{can }2}\,\partial_{ij} w^\beta) - \partial^\beta(h^{ij}_{\text{can }2}\,\partial_{ij} w^\alpha) + f^{\alpha\beta}\,\partial_k(h^{ij}_{\text{can }2}\,\partial_{ij} w^k) + \Delta k_3^{\alpha\beta}. \quad \text{(C 20)}$$

It is a simple matter to verify that the four terms in the right-hand side of (C 20) are 'non-critical'. Indeed critical terms should come from ${}_Y w^0 = 0$ or ${}_Y w^k = \sum_{l \geqslant 1}(\partial_{L-1} r^{-1})\, Y_{kL-1}$ (remember $a = b = d = 0$) so the first three terms cannot contribute to terms of the type $\hat{n}_{ijL_1L_2L_3}$, and the last term, being a 'Laplacian', is inverted without $\lg r$. (However note that *a priori* $\Delta^{-1}(r^B \Delta k_3)_{|B=0} \neq k_3$). Hence $\tilde{C} = \tilde{D} = \tilde{E} = \tilde{F} = 0$. ■

C 2. *Study of the quantity* ${}_{\rm S}w^\alpha$

Assuming for ${}_{\rm S}w_m\,(m < n)$ a structure similar to ${}_{\rm S}h_{\text{part }m-1}$ (equation (C 1)), we have to solve (using (C 1)):

$$\Delta\, {}_{\rm S}w^\alpha_n = \sum_{q,k} G_{Qk} \frac{\hat{n}^Q}{r^{k+2}}, \quad \text{(C 21)}$$

with $k = n - 1 + \sum_{i=1}^n l_i + a + 2b + d$ (notations of (C 3): a is for instance the total number of functions W_P and W'_Q). Then we apply the operator $\Delta^{-1}(r^B \ldots)$ to the right-hand side of (C 21) and find that poles at $B = 0$ arise for $q = k-1$ which implies, similarly to (C 6), $n \leqslant s + 2 - a - 2b - d \leqslant 3$ (since $s = 0, 1$ according to $\alpha = 0, i$). The case $n = 2$ is readily treated using the structure $\sim \partial_P r^{-1} \partial_Q r^{-1}$ of the 'source'; the case $n = 3$ is treated by noticing that the 'spin' index i of eventual critical terms in ${}_{\rm S}w^i_3$ will be carried by one function Y'_L (because of $a = b = d = 0$ and thanks to the form of the sources $\Delta w^\alpha_n \sim h_{n-m}\,\partial w^\alpha_m$ (4.27)): therefore critical terms of the type $\hat{n}_{iL_1L_2L_3}$ cannot appear and we have, for all $n \geqslant 1$,

$$ {}_{\rm S}w^\alpha_n = \sum_{q,k} F_{Qk} \frac{\hat{n}^Q}{r^k}, \quad \text{(C 22)}$$

(with $k \geqslant 1$). ■

Appendix D. Multipolar expansion of the Green function $G_{\rm R}$

The aim of this appendix is to recover by a direct but *formal* calculation the result of theorem 6.1 (i.e. equation (6.5) with (6.4)) using a multipolar expansion of the retarded Green function (with $c = 1$)

$$G_{\rm R}(x' - x) = \delta(t' - t - |\boldsymbol{x}' - \boldsymbol{x}|)/|\boldsymbol{x}' - \boldsymbol{x}| \quad \text{(D 1)}$$

(such that $\Box G_{\rm R}(x'-x) = -4\pi\delta_4(x'-x)$). $G_{\rm R}$ is a function of $r = |\boldsymbol{x}|$, $r' = |\boldsymbol{x}'|$, $t' - t$ and $\boldsymbol{n}\cdot\boldsymbol{n}' = \boldsymbol{x}\cdot\boldsymbol{x}'/(rr')$; it can be expanded in a series of Legendre polynomials of $\boldsymbol{n}\cdot\boldsymbol{n}'$ (Campbell *et al.* 1977). By using (A 26) this series can be written as

$$G_{\rm R}(x'-x) = \frac{Y(t'-t)\, Y(1-|\nu|)}{2rr'} \sum_{q=0}^{+\infty} \frac{(2q+1)!!}{q!} \hat{n}_Q \hat{n}'_Q P_q(\nu), \quad \text{(D 2}a\text{)}$$

where Y is the Heaviside function and

$$\nu = \frac{r'^2 + r^2 - (t'-t)^2}{2rr'}. \quad \text{(D 2}b\text{)}$$

The expansion (D 2) of $G_{\rm R}$ is useful when dealing with the retarded integral $\Box^{-1}_{\rm R}(S(\boldsymbol{x}, t)) = (-1/4\pi)\int \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}t\, G_{\rm R}(x'-x)\, S(\boldsymbol{x}, t)$ where S has a known multipolarity

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Downloaded from rsta.royalsocietypublishing.org

l: $S(\boldsymbol{x}, t) = \hat{n}^L S(r, t-r)$, because we can explicitly perform the integration over the angles. This leads to

$$\Box_R^{-1}(\hat{n}^L S(r, t-r)) = -\frac{\hat{n}'^L}{2r'}\iint \mathrm{d}r\,\mathrm{d}t\, Y(t'-t)\, Y(1-|\nu|)\, rS(r, t-r)\, P_l(\nu) \qquad \text{(D 3)}$$

(with $\hat{n}'^L = n'^{\langle i_1} \ldots n'^{i_l\rangle}$). It is convenient to introduce advanced and retarded variables both for the source point $(r\boldsymbol{n}, t)$ and the field point $(r'\boldsymbol{n}', t')$:

$$u = t-r; \quad u' = t'-r' \qquad \text{(D 4a)}$$

$$v = t+r; \quad v' = t'+r' \qquad \text{(D 4b)}$$

In these variables it is apparent that the domain $\mathscr{D}$ of integration is precisely the (u, v)-projection of the past null cone of $(\boldsymbol{x}', t')$, that is: $\mathscr{D} = \{(u, v); u' \leqslant v \leqslant v' \text{ and } u \leqslant u'\}$. We find

$$\Box_R^{-1}(\hat{n}^L S[\tfrac{1}{2}(v-u), u]) = -\frac{\hat{n}'^L}{4(v'-u')}\iint_{\mathscr{D}} \mathrm{d}u\,\mathrm{d}v (v-u)\, S[\tfrac{1}{2}(v-u), u]\, P_l\left(1-2\frac{(u'-u)\,(v'-v)}{(v'-u')\,(v-u)}\right). \qquad \text{(D 5)}$$

A long but straightforward calculation shows that the following equality holds:

$$P_l\left(1-2\frac{(u'-u)\,(v'-v)}{(v'-u')\,(v-u)}\right) = \frac{(-)^l}{l!}\frac{(v'-u')^{l+1}}{(v-u)^l}\frac{\partial^l}{\partial u'^l}\left\{\frac{(u'-u)^l(u'-v)^l}{(v'-u')^{l+1}}\right\}. \qquad \text{(D 6)}$$

A more elegant way to prove (D 6) is to notice that the function $(u', v'; u, v) \rightarrow P_l(1-2[(u'-u)\,(v'-v)/(v'-u')\,(v-u)])$ is the (local) Riemann function of the self-adjoint Euler–Poisson–Darboux equation $[\partial_{u'v'} + l(l+1)/(v'-u')^2]f = 0$ (equivalent to (2.10)) (see Darboux 1889) that is the only solution which takes the value one along the characteristic lines $u' = u$ and $v' = v$. Therefore the right-hand side of (D 6), which is easily checked to satisfy these properties, is equal to the left-hand side. Now, we use the operator $\hat{\partial}'_L$ (acting on $\boldsymbol{x}'$) and (A 35b) and find (re-establishing the variables r' and t')

$$\Box_R^{-1}(\hat{n}^L S[\tfrac{1}{2}(v-u), u]) = -\frac{1}{8l!}\iint_{\mathscr{D}} \frac{\mathrm{d}u\,\mathrm{d}v}{(v-u)^{l-1}} S[\tfrac{1}{2}(v-u), u]\,\hat{\partial}'_L\left[\frac{(t'-r'-u)^l\,(t'-r'-v)^l}{r'}\right]. \qquad \text{(D 7)}$$

Equation (D 7) is apparently quite different from (6.4). However, these results can be reconciled; indeed plugging (6.7) into (6.4) we obtain

$$\Box_R^{-1}(\hat{n}^L S(r, t-r)) = \frac{1}{2^{l+1}l!}\int_{-\infty}^{t'-r'} \mathrm{d}s\,\hat{\partial}'_L\left\{\int_a^{\frac{1}{2}(t'-r'-s)} \frac{\mathrm{d}x}{x^{l-1}}\frac{(t'-r'-s)^l\,(t'-r'-s-2x)^l}{r'} S(x, s)\right.$$
$$\left. - \int_a^{\frac{1}{2}(t'+r'-s)} \frac{\mathrm{d}x}{x^{l-1}}\frac{(t'+r'-s)^l\,(t'+r'-s-2x)^l}{r'} S(x, s)\right\}. \qquad \text{(D 8)}$$

We can commute the operator $\hat{\partial}'_L$ with the two integral signs $\int_a^{\frac{1}{2}(t'-\epsilon r'-s)}$ (for $\epsilon = \pm 1$) because, thanks to the factor $(t'-\epsilon r'-s-2x)^l$ inside the integrals, all differentiations of the upper limits $\frac{1}{2}(t'-\epsilon r'-s)$ vanish. Then the resulting two integrands: $\hat{\partial}'_L\{(t'-\epsilon r'-s)^l \times (t'-\epsilon r'-s-2x)^l/r'\}\, S(x, s)$ are equal (thanks to (A 36)). We thus obtain

$$\Box_R^{-1}(\hat{n}^L S(r, t-r)) = \frac{-1}{2^{l+1}l!}\int_{-\infty}^{t'-r'} \mathrm{d}s \int_{\frac{1}{2}(t'-r'-s)}^{\frac{1}{2}(t'+r'-s)} \frac{\mathrm{d}x}{x^{l-1}} S(x, s)\,\hat{\partial}'_L\left\{\frac{(t'-\epsilon r'-s)^l\,(t'-\epsilon r'-s-2x)^l}{r'}\right\} \qquad \text{(D 9)}$$

(value independent of ϵ). Equation (D 9) is easily seen to be identical (when $\epsilon = +1$) to the right-hand side of (D 7) with $u = s$ and $v = s + 2x$. ■

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

Appendix E. Some mathematical proofs

E 1. *Proof of lemma* 3.1

Lemma 3.1 will be a corollary of the following lemma.

Lemma E 1. *Let N and K be some non-negative integers and $f(\boldsymbol{x}, t)$ be a function on $\mathbb{R}^n_* \times \mathbb{R}$ (where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_* = \mathbb{R}^n - \{\boldsymbol{0}\}$) satisfying*

(1) $f(\boldsymbol{x}, t) \in C^N(\mathbb{R}^n_* \times \mathbb{R})$;

(2) $\forall t_0, t_1 \in \mathbb{R}$, $\forall m \leqslant N$, $\forall \epsilon > 0$, $\exists d > 0$ *such that* $t_0 \leqslant t \leqslant t_1$ and $0 < |\boldsymbol{x}| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}} < d$ *imply* ($\alpha_1, \ldots, \alpha_m = 0, 1, \ldots, n$, *with* $x_0 \equiv t$)

$$|\partial_{\alpha_1 \ldots \alpha_m} f(\boldsymbol{x}, t)| \leqslant \epsilon |\boldsymbol{x}|^{K-m}. \tag{E 1}$$

Then $f(\boldsymbol{x}, t)$ can be extended by continuity to a function on $\mathbb{R}^{n+1}$, which is $C^{N'}(\mathbb{R}^{n+1})$ with $N' = \inf(N, K)$. Moreover, $\forall m \leqslant N'$, $\forall t \in \mathbb{R}$, we have:

$$\partial_{\alpha_1 \cdots \alpha_m} f(\boldsymbol{0}, t) = 0. \tag{E 2}$$

Proof. Because $f(\boldsymbol{x}, t)$ is at least $C^0(\mathbb{R}^n_* \times \mathbb{R})$, we see from (E 1) with $m = 0$ that $f(\boldsymbol{x}, t)$, when extended to $\mathbb{R}^{n+1}$ by $f(\boldsymbol{0}, t) = 0$, belongs to $C^0(\mathbb{R}^{n+1})$. Suppose $N \geqslant 1$ and $K \geqslant 1$. Then (E 1), when $m = 0$, implies that $f(\boldsymbol{x}, t)$ is differentiable in $(\boldsymbol{0}, t)$ with $\partial_\alpha f(\boldsymbol{0}, t) = 0$, and, when $m = 1$, that $\partial_\alpha f(\boldsymbol{x}, t)$ is continuous in $(\boldsymbol{0}, t)$. Thus: $f(\boldsymbol{x}, t) \in C^1(\mathbb{R}^{n+1})$ with $f(\boldsymbol{0}, t) = \partial_\alpha f(\boldsymbol{0}, t) = 0$. By the same reasoning we have $f(\boldsymbol{x}, t) \in C^p(\mathbb{R}^{n+1})$ with $f(\boldsymbol{0}, t) = \ldots = \partial_{\alpha_1 \ldots \alpha_p} f(\boldsymbol{0}, t) = 0$ for all p less than N and K. Hence the result. ∎

Now, if $f(\boldsymbol{x}, t)$ satisfies the hypotheses of lemma 3.1, then, by lemma E 1 with $n = 3$, we have: $\forall q \in \mathbb{N}, f^{(q)}(\boldsymbol{x}, t) \in C^{N'}(\mathbb{R}^4)$. This proves lemma 3.1.

E 2. *Proof of lemma* 3.3

Let us write the retarded integral (3.4) of $f(\boldsymbol{x}, t) \in O^N(r^N)$ under the form

$$(\Box^{-1}_{\mathrm{R}} f)(\boldsymbol{x}', t') = -\frac{1}{4\pi} \int \mathrm{d}\rho \, \mathrm{d}\Theta \, \mathrm{d}\Phi \, \rho \sin \Theta \, f(\boldsymbol{x}' + \rho \boldsymbol{n}, t' - \rho) \tag{E 3}$$

(with $\boldsymbol{x} - \boldsymbol{x}' = \rho \boldsymbol{n} = \rho(\sin\Theta \cos\Phi, \sin\Theta \sin\Phi, \cos\Theta)$). Then, by a standard theorem for integrals with compact support, such as (E 3), we have (with primes suppressed)

$$\forall q \in \mathbb{N}, \quad (\partial/\partial t)^q (\Box^{-1}_{\mathrm{R}} f)(\boldsymbol{x}, t) \in C^N(\mathbb{R}^4) \tag{E 4}$$

and we can differentiate under the sign $\Box^{-1}_{\mathrm{R}}$. We apply Taylor's formula with integral remainder to $(\Box^{-1}_{\mathrm{R}} f)(\boldsymbol{x}, t)$ between the points $(\boldsymbol{x}, t)$ and $(\boldsymbol{0}, t)$, up to order N:

$$\Box^{-1}_{\mathrm{R}} f - \sum_{l=0}^{N-1} x^{i_1} \ldots x^{i_l} F_{i_1 \ldots i_l}(t) = x^{i_1} \ldots x^{i_N} \int_0^1 \mathrm{d}\alpha \frac{(1-\alpha)^{N-1}}{(N-1)!} \left(\frac{\partial^N}{\partial x^{i_1} \ldots \partial x^{i_N}} \Box^{-1}_{\mathrm{R}} f \right)(\alpha \boldsymbol{x}, t) \tag{E 5}$$

with

$$F_{i_1 \ldots i_l}(t) = \frac{1}{l!} \left(\frac{\partial^l}{\partial x^{i_1} \ldots \partial x^{i_l}} \Box^{-1}_{\mathrm{R}} f \right)(\boldsymbol{0}, t). \tag{E 6}$$

From (E 4) we check that $F_{i_1 \ldots i_l}(t)$ is zero for $t \leqslant -T$ and $C^\infty(\mathbb{R})$. Hence, applying $(\partial/\partial t)^q$ to the left-hand side of (E 5) leads to a function which is zero for $t \leqslant -T$ and $C^N(\mathbb{R}^4)$. Therefore the right-hand side of (E 5) satisfies the defining properties (*a*) and (*b*) of the $O^N(r^N)$ class (definition 3.1). Requirement (*c*) follows from the continuity of $(\partial^N / \partial x^{i_1} \ldots \partial x^{i_N})(\partial/\partial t)^q \Box^{-1}_{\mathrm{R}} f$, which provides for this right-hand side the desired bound $M |x^{i_1} \ldots x^{i_l}| \leqslant M r^N$. ∎

Downloaded from rsta.royalsocietypublishing.org

Acknowledgements

It is a pleasure to acknowledge clarifying discussions with J. Ehlers and B. Schmidt at an early state of this work. The authors thank K. S. Thorne for numerous fruitful discussions. Many useful discussions with the members of the Caltech Paradigm Society are gratefully acknowledged. T.D. thanks B. Simon for an informative discussion. The authors thank K. S. Thorne for his kind invitations to visit the Theoretical Astrophysics group at Caltech, and for financial support which made these visits possible. This work was supported in part by the C.N.R.S., the 'Ecole Polytechnique', the 'Fondation de France', the 'Ministère des Relations Extérieures', and by the National Science Foundation under grant no. AST 82-14126.

References

Anderson, J. L. 1984 *J. math. Phys.* **25**, 1947.
Anderson, J. L., Kates, R. E., Kegeles, L. S. & Madonna, R. G. 1982 *Phys. Rev.* D **25**, 2038.
Ashtekar, A. 1984 In *General relativity and gravitation* (*Proceedings of GR*10) (ed. B. Bertotti, *et al.*) Dordrecht: Reidel.
Ashtekar, A. & Dray, T. 1981 *Communs math. Phys.* **79**, 581.
Bardeen, J. M. & Press, W. H. 1973 *J. math. Phys.* **14**, 7.
Beig, R. 1981 *Acta phys. austriaca* **53**, 249.
Beig, R. & Simon, W. 1981 *Proc. R. Soc. Lond.* A **376**, 333.
Bel, L., Damour, T., Deruelle, N., Ibañez, J. & Martin, J. 1981 *Gen. Rel. Grav.* **13**, 963.
Bičak, J., Hoenselaers, C. & Schmidt, B. G. 1983 *Proc. R. Soc. Lond.* A**390**, 397, 411.
Blanchet, L. 1984 Thèse de troisième cycle, Université de Paris VI.
Blanchet, L. & Damour, T. 1984*a* *C.r. Acad. Sci., Paris* (II) **298**, 431.
Blanchet, L. & Damour, T. 1984*b* *Phys. Lett.* **104** A, 82.
Bondi, H., van der Burg, M. G. J. & Metzner, A. W. K. 1962 *Proc. R. Soc. Lond.* A **269**, 21.
Bonnor, W. B. 1959 *Phil. Trans. R. Soc. Lond.* A **251**, 233.
Bonnor, W. B. 1974 In *Ondes et radiations gravitationnelles*, CNRS, Paris, p. 73.
Bonnor, W. B. & Rotenberg, M. A. 1966 *Proc. R. Soc. Lond.* A **289**, 247.
Campbell, W. B., Macek, J. & Morgan, T. A. 1977 *Phys. Rev.* D **15**, 2156.
Coope, J. A. R. 1970 *J. math. Phys.* **11**, 1591.
Coope, J. A. R. & Snider, R. F. 1970 *J. math. Phys.* **11**, 1003.
Coope, J. A. R., Snider, R. F. & McCourt, F. R. 1965 *J. chem. Phys.* **43**, 2269.
Copson, E. T. 1975 *Partial differential equations*, §6.6. Cambridge University Press.
Couch, W. E., Torrence, R. J., Janis, A. I. & Newman, E. T. 1968 *J. math. Phys.* **9**, 484.
Courant, R. & Hilbert, D. 1953 *Methods of mathematical physics*, vol. I. New York: Interscience.
Damour, T. 1983*a* In *Gravitational radiation* (ed. N. Deruelle & T. Piran), p. 59. Amsterdam: North Holland.
Damour, T. 1983*b* *Phys. Rev. Lett.* **51**, 1019.
Damour, T. 1985 In *Proceedings of the Fourth Marcel Grossmann Meeting*, Rome, June 1985, ed. R. Ruffini. North-Holland. (To be published.)
Darboux, G. 1889 *Leçons sur la théorie générale des surfaces*, part 2 (*Les congruences et les équations linéaires aux dérivées partielles. Les lignes tracées sur les surfaces*), book 4, chapters 3 and 4. Paris: Gauthier-Villars. (Reprinted by Chelsea Publishing Company, New York, 1972.)
Ehlers, J. 1984 In *Proceedings of the 7th International Congress of Logic, Methodology and Philosophy of Science*. North Holland.
Einstein, A. 1916 *Sber. preuss. Akad. Wiss. Berl.* p. 688.
Einstein, A. 1918 *Sber. preuss. Akad. Wiss. Berl.* p. 154.
Einstein, A., Infeld, L. & Hoffmann, B. 1938 *Ann. Math.* **39**, 65.
Epstein, R. & Wagoner, R. V. 1975 *Astrophys. J.* **197**, 717.
Finn, L. S. 1985 *Classical and quantum gravity* **2**, 380.
Fock, V. A. 1959 *Theory of space, time and gravitation*. London: Pergamon.
Friedrich, H. 1983*a* In *Gravitational radiation* (ed. N. Deruelle & T. Piran), p. 407. Amsterdam: North Holland.
Friedrich, H. 1983*b* *Communs math. Phys.* **91**, 445.
Futamase, T. 1983 *Phys. Rev.* D **28**, 2373.
Futamase, T. & Schutz, B. F. 1983 *Phys. Rev.* D **28**, 2363.
Gel'fand, I. M., Minlos, R. A. & Shapiro, Z. Ya. 1963 *Representations of the rotation and Lorentz groups*. Oxford: Pergamon.
Geroch, R. 1970 *J. math. Phys.* **11**, 2580.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Downloaded from rsta.royalsocietypublishing.org

Geroch, R. 1977 In *Asymptotic structure of space-time* (ed. F. P. Esposito & L. Witten), p. 1. New York: Plenum Press.
Gürsel, Y. 1983 *Gen. Rel. Grav.* **15**, 737.
Hadamard, J. 1932 *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*. Paris: Hermann.
Hansen, R. O. 1974 *J. math. Phys.* **15**, 46.
Hunter, A. J. & Rotenberg, M. A. 1969 *J. Phys.* A **2**, 34.
Infeld, L. & Plebanski, J. 1960 *Motion and relativity*. Oxford: Pergamon Press.
Isaacson, R. A. & Winicour, J. 1968 *Phys. Rev.* **168**, 1451.
Jackson, J. D. 1975 *Classical electrodynamics*. (Second edition.) New York: John Wiley & Sons.
Kerlick, G. D. 1980 *Gen. Rel. Grav.* **12**, 467 and 521.
Kovács, S. & Thorne, K. S. 1977 *Astrophys. J.* **217**, 252 and references therein.
Lucquiaud, J. C. 1984*a* *C.r. Acad. Sci., Paris* (I) **299**, 467.
Lucquiaud, J. C. 1984*b* *J. math. pures appl.* **63**, 265.
Madore, J. 1970 *Ann. Inst. H. Poincaré* **12**, 365.
Newman, E. & Penrose, R. 1962 *J. math. Phys.* **3**, 566.
Penrose, R. 1963 *Phys. Rev. Lett.* **10**, 66.
Penrose, R. 1965 *Proc. R. Soc. Lond.* A **284**, 159.
Persides, S. 1971 *Astrophys. J.* **170**, 479.
Pirani, F. A. E. 1962*a* In *Gravitation: an introduction to current research* (ed. L. Witten). New York: Wiley.
Pirani, F. A. E. 1962*b* In *Recent developments in general relativity*. Oxford: Pergamon. (Infeld Festschrift.)
Pirani, F. A. E. 1964 In *Lectures on general relativity*, eds A. Trautman, F. A. E. Pirani & H. Bondi, p. 249. Englewood Cliffs: Prentice-Hall.
Porill, J. & Stewart, J. M. 1981 *Proc. R. Soc. Lond.* A **376**, 451.
Riesz, M. 1938 *Bull. Société Math. France*, p. 66.
Sachs, R. 1961 *Proc. R. Soc. Lond.* A **264**, 309.
Sach, R. K. 1962 *Proc. R. Soc. Lond.* A **270**, 103.
Sachs, R. & Bergmann, P. G. 1958 *Phys. Rev.* **112**, 674.
Schäfer, G. 1985 *Ann. Phys.* **161**, 81.
Schmidt, B. G. 1979 In *Isolated gravitating systems in general relativity* (ed. J. Ehlers), p. 11. Amsterdam: North Holland.
Schmidt, B. G. 1981 *Communs math. Phys.* **78**, 447.
Schmidt, B. G. & Stewart, J. M. 1979 *Proc. R. Soc. Lond.* A **367**, 503.
Schumaker, B. L. & Thorne, K. S. 1983 *Mon. Not. R. astr. Soc.* **203**, 457.
Simon, W. & Beig, R. 1983 *J. math. Phys.* **24**, 1163.
Thorne, K. S. 1977 In *Topics in theoretical and experimental gravitation physics* (ed. V. De Sabbata & J. Weber.) London: Plenum Press.
Thorne, K. S. 1980 *Rev. mod. Phys.* **52**, 299.
Thorne, K. S. 1983 In *Gravitational radiation* (ed. N. Deruelle & T. Piran), p. 1. Amsterdam: North Holland.
Trautman, A. 1958 *Bull. Pol. Acad. Sci.* III, **6**, 403, 407.
Walker, M. & Will, C. M. 1979 *Phys. Rev.* D **19**, 3495.
Winicour, J. 1983 *J. math. phys.* **24**, 1193.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES